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Monodromy of the trigonometric Casimir connection for $\mathfrak{sl}_2$

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Abstract. We show that the monodromy of the trigonometric Casimir connection on the tensor product of evaluation modules of the Yangian $Y_h\mathfrak{sl}_2$ is described by the quantum Weyl group operators of the quantum loop algebra $U_\hbar(L\mathfrak{sl}_2)$. The proof is patterned on the second author's computation of the monodromy of the rational Casimir connection for $\mathfrak{sl}_n$ via the dual pair $(\mathfrak{gl}_k, \mathfrak{gl}_n)$, and rests ultimately on the Etingof–Geer–Schiffmann computation of the monodromy of the trigonometric KZ equations. It relies on two new ingredients: an affine extension of the duality between the R–matrix of $U_h\mathfrak{sl}_k$ and the quantum Weyl group element of $U_\hbar L\mathfrak{sl}_2$, and a formula expressing the quantum Weyl group action of the coroot lattice of SL_2 in terms of the commuting generators of $U_\hbar(L\mathfrak{sl}_2)$. Using this formula, we define quantum Weyl group operators for the quantum loop algebra $U_\hbar(L\mathfrak{gl}_2)$, and show that they describe the monodromy of the trigonometric Casimir connection on a tensor product of evaluation modules of the Yangian $Y_\mathsf{h}\mathfrak{gl}_2$.

1. Introduction

1.1. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, G the corresponding connected and simply–connected Lie group, $H \subset G$ a maximal torus and W the corresponding Weyl group. In [**28**], a flat W–equivariant connection $\widehat{\nabla}_C$ was constructed on H which has logarithmic singularities on the root subtori of H and values in any finite–dimensional representation of the Yangian $Y_\mathsf{h}\mathfrak{g}$. By analogy with the description of the monodromy of the rational Casimir connection obtained in [**26, 27**], it was conjectured in [**28**] that the monodromy of the trigonometric Casimir connection $\widehat{\nabla}_C$ is described by the action of the affine braid group B_G of G arising from the quantum Weyl group operators of the quantum loop algebra $U_\hbar(L\mathfrak{g})$.

1.2. The aim of the present paper is to prove this conjecture when $\mathfrak{g} = \mathfrak{sl}_2$ and V is a tensor product of evaluation modules. Note that, by a theorem of Chari–Pressley [**5**], such representations include all irreducible $Y_\mathsf{h}\mathfrak{sl}_2$–modules. To state our main result, let $V_1, \ldots, V_k$ be finite–dimensional $\mathfrak{sl}_2$–modules, $z_1, \ldots, z_k$ points in $\mathbb{C}$, and

$$V(\underline{z}) = V_1(z_1) \otimes \cdots \otimes V_k(z_k)$$

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the tensor product of the corresponding evaluation representations of $Y_{\mathsf{h}}\mathfrak{sl}_2$. The monodromy of the trigonometric Casimir connection yields an action of the affine braid group B_{SL_2} on $V(\underline{z})$.

Let $\mathcal{V}_i$ be a quantum deformation of V_i, that is a module over the quantum group $U_\hbar\mathfrak{sl}_2$ such that $\mathcal{V}_i/\hbar\mathcal{V}_i \cong V_i$. Set $\hbar = 4\pi\imath\mathsf{h}$ and $\zeta_i = \exp(-\hbar z_a)$, and consider the tensor product of evaluation representations of the quantum loop algebra $U_\hbar(L\mathfrak{sl}_2)$ given by

$$\mathcal{V}(\underline{\zeta}) = \mathcal{V}_1(\zeta_1) \otimes \cdots \otimes \mathcal{V}_k(\zeta_k)$$

The quantum Weyl group operators $\mathbb{S}_0, \mathbb{S}_1$ of $U_\hbar(L\mathfrak{sl}_2)$ yield a representation of B_{SL_2} on $\mathcal{V}(\underline{\zeta})$ [**19, 20, 24**]. The main result of this paper is the following

THEOREM. *The monodromy action of the affine braid group B_{SL_2} on $V(\underline{z})$ is equivalent to its quantum Weyl group action on $\mathcal{V}(\underline{\zeta})$.*

1.3. The proof of the above theorem relies on two dualities between the Lie algebras $\mathfrak{sl}_k$ and $\mathfrak{sl}_n$ discovered in [**26**][1]. The first duality arises from their joint action on the space $\mathbb{C}[\mathcal{M}_{k,n}]$ of functions on $k \times n$ matrices, and identifies the rational Casimir connection of $\mathfrak{sl}_k$ with the rational KZ connection on n points for $\mathfrak{sl}_k$. The second duality arises from the action of the corresponding quantum groups $U_\hbar\mathfrak{sl}_k$ and $U_\hbar\mathfrak{sl}_n$ on a noncommutative deformation of $\mathbb{C}[\mathcal{M}_{k,n}]$, and identifies the quantum Weyl group elements of $U_\hbar\mathfrak{sl}_n$ with the R–matrices of $U_\hbar\mathfrak{sl}_k$.

These dualities were used in [**26**] together with the Kohno–Drinfeld theorem for $\mathfrak{sl}_k$, to show that the monodromy of the rational Casimir connection of $\mathfrak{sl}_n$ is described by the quantum Weyl group operators of $U_\hbar\mathfrak{sl}_n$.

1.4. In this paper, we apply a similar strategy to compute the monodromy of the trigonometric Casimir connection of $\mathfrak{sl}_2$ and, in fact, $\mathfrak{gl}_2$. The latter connection is an extension of the former to the maximal torus of GL_2 constructed in [**28**], and takes values in the Yangian $Y_{\mathsf{h}}\mathfrak{gl}_2$. Its evaluation on a tensor product of evaluation modules coincides, up to abelian terms, with the trigonometric dynamical differential equations considered in [**25**]. In particular, we also compute the monodromy of these equations.

The duality between the Casimir and KZ connections identifies the trigonometric Casimir connection of $\mathfrak{gl}_2$ with the trigonometric KZ connection of $\mathfrak{gl}_k$ (see, *e.g.,* [**25**]). In turn, the monodromy of the latter was computed by Etingof–Geer–Schiffmann in terms of data coming from the quantum group $U_\hbar\mathfrak{gl}_k$ [**12**]. This reduces the original problem to interpreting this data in terms of the quantum loop algebra $U_\hbar(L\mathfrak{gl}_2)$.

Part of this interpretation, namely the one pertaining to the data describing the monodromy of the finite braid group $\mathbb{Z} \cong B_{\mathfrak{sl}_2} \subset B_{SL_2}$, is provided by the duality between $U_\hbar\mathfrak{gl}_k$ and $U_\hbar\mathfrak{gl}_2$ of [**26**] alluded to in 1.3. What remains is the description of the operators giving the action of the coroot lattice $\mathbb{Z}^2 \cong Q^\vee \subset B_{GL_2}$ of GL_2, in terms of appropriate, commuting quantum Weyl group operators of $U_\hbar(L\mathfrak{gl}_2)$.

1.5. To the best of our knowledge, quantum Weyl group operators giving an action of the coroot lattice of GL_2 on finite–dimensional representations of the quantum loop algebra $U_\hbar(L\mathfrak{gl}_2)$ have not been defined. Moreover, for $U_\hbar(L\mathfrak{sl}_2)$, no compact, explicit formula appears to be known for the element $\mathbb{S}_0\mathbb{S}_1$ giving the

[1] in the case relevant to the present paper, $n = 2$.

action of the generator of the coroot lattice of SL_2. In this paper, we give the following solution to both of these problems.

Let $\mathfrak{t} \subset \mathfrak{gl}_2$ and $\mathfrak{h} \subset \mathfrak{sl}_2$ be the Cartan subalgebras of diagonal and traceless diagonal matrices respectively, and $U_0 \subset U_\hbar(L\mathfrak{gl}_2)$, $U_0' \subset U_\hbar(L\mathfrak{sl}_2)$ the commutative subalgebras deforming $U(\mathfrak{t}[z,z^{-1}])$ and $U(\mathfrak{h}[z,z^{-1}])$. Then, we prove the following.f

THEOREM.

(1) *There exist elements* $\mathbb{L}_1, \mathbb{L}_2$ *in a completion of* U_0 *such that* $\{\mathbb{S} = \mathbb{S}_1, \mathbb{L}_1, \mathbb{L}_2\}$ *satisfy the defining relations of the affine braid group* B_{GL_2}.

(2) *The element* $\mathbb{L} = \mathbb{L}_1\mathbb{L}_2^{-1}$ *lies in a completion of* U_0', *and coincides with the quantum Weyl group element* $\mathbb{S}_0\mathbb{S}_1$ *giving the action of the generator of the coroot lattice of* SL_2.

The elements $\mathbb{L}_1, \mathbb{L}_2$ are given by explicit formulae in terms of the generators of U_0. For $\mathbb{L} = \mathbb{L}_1\mathbb{L}_2^{-1}$, these are as follows. Let $\{H_k\}_{k\in\mathbb{Z}}$ be the generators of U_0' with classical limit $\{h \otimes z^k\}$, where h is the standard generator of $\mathfrak{h}$ (see Section 8). Define, for any $r \in \mathbb{N}$,

$$\widetilde{H}_r = H_0 \quad + \sum_{s=1}^{r} (-1)^s \begin{pmatrix} r \\ s \end{pmatrix} \frac{s}{[s]} H_s$$

and note that $\widetilde{H}_r = h \otimes (1-z)^r \bmod \hbar$. Then, we show that

$$\mathbb{L} = \exp\left(\sum_{r\geq 1} \frac{\widetilde{H}_r}{r} \right)$$

thus extending to the q–setting the fact that the classical limit of $\mathbb{L}$ is the loop

$$z \mapsto \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = \exp(-h \log z)$$

The operators $\mathbb{L}_1, \mathbb{L}_2$ are given by similar formulae. These generalise in fact to any complex semisimple Lie algebra and to $\mathfrak{gl}_n$ [**17**].

1.6. Once the operators $\mathbb{L}_1, \mathbb{L}_2$ are explicitly defined, a direct computation shows that their action on quantum $k \times 2$ matrix space coincides with that of the $U_\hbar\mathfrak{gl}_k$ operators which, by [**12**] describe the monodromy of the trigonometric KZ connection of $\mathfrak{gl}_k$, thus providing an extension of the q–duality of [**26**] to the affine setting. Theorem 1.2, and its analogue for $\mathfrak{gl}_2$ follow as a direct consequence.

1.7. The results of the present paper extend without essential modification to the case of $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{gl}_n$, and give a computation of the monodromy of the trigonometric Casimir connection of $\mathfrak{g}$ with values in a tensor product of arbitrary finite–dimensional evaluation representations of the Yangian $Y_\mathfrak{h}\mathfrak{g}$, in terms of the quantum Weyl group operators of the quantum loop algebra $U_\hbar(L\mathfrak{g})$.

1.8. Outline of the paper. Sections 2 and 3 review the definition of the Yangian and trigonometric Casimir connections of the Lie algebras $\mathfrak{sl}_2$ and $\mathfrak{gl}_2$ respectively. Section 4 gives presentations of the affine braid groups B_{SL_2} and B_{GL_2}, and describes the embedding $B_{SL_2} \subset B_{GL_2}$ resulting from the inclusion of the maximal tori of SL_2 and GL_2 in terms of the corresponding generators.

In Section 5, we review the definition of the trigonometric KZ connection for the Lie algebra $\mathfrak{gl}_k$ and, in Section 6 the fact that, under $(\mathfrak{gl}_k, \mathfrak{gl}_2)$–duality, the trigonometric Casimir connection for $\mathfrak{gl}_2$ is identified with the trigonometric KZ connection for $\mathfrak{gl}_k$. In Section 7 we describe, following [**12**], the monodromy of the latter connection in terms of the quantum group $U_\hbar\mathfrak{gl}_k$.

In Section 8, we review the definition of the quantum loop algebras $U_\hbar(L\mathfrak{gl}_2)$ and $U_\hbar(L\mathfrak{sl}_2)$. Section 9 contains the main construction of this paper. We first extend the quantum Weyl group action of the affine braid group B_{SL_2} on $U_\hbar(L\mathfrak{sl}_2)$ to one of B_{GL_2} on $U_\hbar(L\mathfrak{gl}_2)$. We then show that this action is essentially inner, by exhibiting elements in an appropriate completion of the maximal commutative subalgebra of $U_\hbar(L\mathfrak{gl}_2)$, whose adjoint action coincides with the quantum Weyl group action of the coroot lattice of $\mathfrak{gl}_2$.

Section 10 describes the joint action of $U_\hbar\mathfrak{gl}_k$ and $U_\hbar\mathfrak{gl}_2$ on the space $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$ of quantum $k\times 2$ matrices. In Section 11, we prove the equality of two actions of the affine braid group B_{GL_2} on $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$. The first arises from its structure as $U_\hbar\mathfrak{gl}_k$–module, and describes the monodromy of the trigonometric KZ equations; the second from its structure as a tensor product of k evaluation modules of $U_\hbar(L\mathfrak{gl}_2)$.

In Section 12, we prove that the monodromy of the trigonometric Casimir connection for $\mathfrak{g}=\mathfrak{sl}_2$ (resp. $\mathfrak{g}=\mathfrak{gl}_2$) on a tensor product of evaluation modules is described by the quantum Weyl group operators of $U_\hbar(L\mathfrak{g})$.

Appendix A outlines the computation of the monodromy of the trigonometric KZ connection given in [**12**]. Appendix B contains the proof of a technical result bearing upon the completions of the quantum loop algebras $U_\hbar(L\mathfrak{sl}_2)$ and $U_\hbar(L\mathfrak{gl}_2)$ required to handle quantum Weyl group elements.

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2. The trigonometric Casimir connection of $\mathfrak{sl}_2$

2.1. The Yangian $Y_{\mathsf{h}}\mathfrak{sl}_2$ [10]. The Yangian $Y_{\mathsf{h}}\mathfrak{sl}_2$ is the unital, associative algebra over $\mathbb{C}[\mathsf{h}]$ generated by elements $\{\xi_r, e_r, f_r\}_{r\in\mathbb{N}}$, subject to the relations

(Y1) For each $r,s\in\mathbb{N}$,
$$[\xi_r,\xi_s]=0$$

(Y2) For each $r\in\mathbb{N}$,
$$[\xi_0,e_r]=2e_r \qquad \text{and} \qquad [\xi_0,f_r]=-2f_r$$

(Y3) For each $r,s\in\mathbb{N}$,
$$[e_r,f_s]=\xi_{r+s}$$

(Y4) For each $r,s\in\mathbb{N}$,
$$\begin{aligned}[\xi_{r+1},e_s]-[\xi_r,e_{s+1}] &= \mathsf{h}\,(\xi_r e_s+e_s\xi_r)\\ [\xi_{r+1},f_s]-[\xi_r,f_{s+1}] &= -\mathsf{h}\,(\xi_r f_s+f_s\xi_r)\end{aligned}$$

(Y5) For each $r,s\in\mathbb{N}$,
$$\begin{aligned}[e_{r+1},e_s]-[e_r,e_{s+1}] &= \mathsf{h}\,(e_r e_s+e_s e_r)\\ [f_{r+1},f_s]-[f_r,f_{s+1}] &= -\mathsf{h}\,(f_r f_s+f_s f_r)\end{aligned}$$

$Y_\mathsf{h}\mathfrak{sl}_2$ is an $\mathbb{N}$–graded algebra with $\deg(x_r) = r$ and $\deg \mathsf{h} = 1$. Moreover, it is a Hopf algebra with coproduct determined by

$$\Delta(x_0) = x_0 \otimes 1 + 1 \otimes x_0$$

for $x = e, f, \xi$, and

$$\Delta(\xi_1) = \xi_1 \otimes 1 + 1 \otimes \xi_1 + \mathsf{h}\,(\xi_0 \otimes \xi_0 - 2f_0 \otimes e_0) \tag{2.1}$$

Let $\{e, f, h\}$ be the standard basis of the Lie algebra $\mathfrak{sl}_2$. Then, the map

$$e \to e_0 \qquad f \to f_0 \qquad h \to \xi_0$$

defines an embedding of $\mathfrak{sl}_2$ into $Y_\mathsf{h}\mathfrak{sl}_2$. In particular, $Y_\mathsf{h}\mathfrak{sl}_2$ is acted upon by $\mathfrak{sl}_2$ via the adjoint action. This action is integrable since the graded components of $Y_\mathsf{h}\mathfrak{sl}_2$ are finite–dimensional.

2.2. The trigonometric Casimir connection [28]. Let $G = SL_2(\mathbb{C})$, $H \subset G$ the maximal torus consisting of diagonal matrices and $\mathfrak{h} \subset \mathfrak{sl}_2$ its Lie algebra. The Weyl group $W \cong \mathbb{Z}_2$ of G acts on H and on the centraliser of $\mathfrak{h}$ in $Y_\mathsf{h}\mathfrak{sl}_2$.

The trigonometric Casimir connection of $\mathfrak{sl}_2$ is the flat, W–equivariant connection on H with values in $Y_\mathsf{h}\mathfrak{sl}_2$ given by

$$\widehat{\nabla}_C^{\mathfrak{sl}_2} = d - \left(\frac{\mathsf{h}\kappa}{e^\alpha - 1} - t_1\right) d\alpha$$

where $\kappa = e_0 f_0 + f_0 e_0$ is the truncated Casimir element of $\mathfrak{sl}_2$, $\alpha \in \mathfrak{h}^*$ is defined by $\alpha(h) = 2$, $d\alpha$ is the corresponding translation–invariant one–form on H, and

$$t_1 = \xi_1 - \frac{\mathsf{h}}{2}\xi_0^2$$

2.3. Evaluation homomorphism. For any $s \in \mathbb{C}[\mathsf{h}]$, there is an algebra homomorphism $\mathrm{ev}_s : Y_\mathsf{h}\mathfrak{sl}_2 \to U\mathfrak{sl}_2[\mathsf{h}]$ which is equal to the identity on $\mathfrak{sl}_2 \subset Y_\mathsf{h}\mathfrak{sl}_2$ and is otherwise determined by [**5**, Prop. 2.5]

$$t_1 \mapsto sh - \frac{\mathsf{h}}{2}\kappa$$

Note that if $s \in \mathsf{h}\mathbb{C}[\mathsf{h}]$, ev_s maps elements of positive degree in $Y_\mathsf{h}\mathfrak{sl}_2$ to $\mathsf{h}U\mathfrak{sl}_2[\mathsf{h}]$ and therefore extends to a homomorphism $\widehat{Y_\mathsf{h}\mathfrak{sl}_2} \to U\mathfrak{sl}_2[[\mathsf{h}]]$, where $\widehat{Y_\mathsf{h}\mathfrak{sl}_2}$ is the completion of $Y_\mathsf{h}\mathfrak{sl}_2$ with respect to its grading.

2.4. Let $k \in \mathbb{N}^*$, $\underline{s} = (s_1, \dots, s_k) \in \mathbb{C}[\mathsf{h}]^k$ and consider the homomorphism

$$\mathrm{ev}_{\underline{s}} = \mathrm{ev}_{s_1} \otimes \cdots \otimes \mathrm{ev}_{s_k} \circ \Delta^{(k)} : Y_\mathsf{h}\mathfrak{sl}_2 \to U\mathfrak{sl}_2^{\otimes k}[\mathsf{h}]$$

where $\Delta^{(k)} : Y_\mathsf{h}\mathfrak{sl}_2 \to Y_\mathsf{h}\mathfrak{sl}_2^{\otimes k}$ is the iterated coproduct.

PROPOSITION. *The image of* $\widehat{\nabla}_C^{\mathfrak{sl}_2}$ *under the homomorphism* $\mathrm{ev}_{\underline{s}}$ *is the* $U\mathfrak{sl}_2^{\otimes k}[\mathsf{h}]$–*valued connection on* H *given by*

$$\widehat{\nabla}_{C,\underline{s}}^{\mathfrak{sl}_2} = d - \left(\mathsf{h}\frac{\Delta^{(k)}(\kappa)}{e^\alpha - 1} - A\right) d\alpha$$

where

$$A = \sum_{a=1}^{k} s_a h^{(a)} - \frac{\mathsf{h}}{2}\sum_{a=1}^{k} \kappa^{(a)} - 2\mathsf{h} \sum_{1 \le a < b \le k} f^{(a)} e^{(b)}$$

and, for any $x \in U\mathfrak{sl}_2$, $x^{(a)} = 1^{\otimes(a-1)} \otimes x \otimes 1^{\otimes(k-a)}$.

PROOF. The element t_1 defined in §2.2 satisfies $\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1 - 2\mathsf{h} f_0 \otimes e_0$, so that

$$\Delta^{(k)}(t_1) = \sum_a t_1^{(a)} - 2\mathsf{h} \sum_{a<b} f_0^{(a)} e_0^{(b)}$$

The result follows since $\mathrm{ev}_s(t_1) = sh - \frac{\mathsf{h}}{2}\kappa$. □

3. The trigonometric Casimir connection of $\mathfrak{gl}_2$

3.1. The Yangian $Y_\mathsf{h}\mathfrak{gl}_2$ [9]. $Y_\mathsf{h}\mathfrak{gl}_2$ is the unital, associative algebra over $\mathbb{C}[\mathsf{h}]$ generated by elements $\{t_{ij}^{(r)}\}_{1\le i,j\le 2, r\ge 1}$, subject to the relations[2]

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = \mathsf{h}\left(t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}\right)$$

for any $r, s \ge 0$, where $t_{ij}^{(0)} = \mathsf{h}^{-1}\delta_{ij}$. These imply that $E_{ij} \mapsto t_{ij}^{(1)}$ gives an embedding of $\mathfrak{gl}_2$ into $Y_\mathsf{h}\mathfrak{gl}_2$, and that $Y_\mathsf{h}\mathfrak{gl}_2$ is an $\mathbb{N}$–graded algebra with

$$\deg(t_{ij}^{(r)}) = r - 1 \qquad \text{and} \qquad \deg(\mathsf{h}) = 1$$

Moreover, $Y_\mathsf{h}\mathfrak{gl}_2$ is a Hopf algebra with coproduct given by

$$\Delta(t_{ij}(u)) = \sum_k t_{ik}(u) \otimes t_{kj}(u)$$

where $t_{ij}(u) = \mathsf{h}\sum_{r\ge 0} t_{ij}^{(r)} u^{-r}$.

3.2. The embedding $Y_\mathsf{h}\mathfrak{sl}_2 \subset Y_\mathsf{h}\mathfrak{gl}_2$ [4], [22]. Let $e(u), f(u), \xi(u) \in Y_\mathsf{h}\mathfrak{sl}_2[[u^{-1}]]$ be the generating series

$$e(u) = \mathsf{h}\sum_{r\ge 0} e_r u^{-r-1} \qquad f(u) = \mathsf{h}\sum_{r\ge 0} f_r u^{-r-1} \qquad \xi(u) = 1 + \mathsf{h}\sum_{r\ge 0} \xi_r u^{-r-1}$$

Then, the following defines an embedding of graded Hopf algebras $\imath : Y_\mathsf{h}\mathfrak{sl}_2 \to Y_\mathsf{h}\mathfrak{gl}_2$ [**22**, Rem. 3.1.8]

$$e(u) \mapsto t_{21}(u) t_{11}(u)^{-1} \qquad f(u) \mapsto t_{11}(u)^{-1} t_{12}(u)$$

$$\xi(u) \mapsto t_{22}(u) t_{11}(u)^{-1} - t_{21}(u) t_{11}(u)^{-1} t_{12}(u) t_{11}(u)^{-1}$$

In particular,

$$\imath(e_0) = t_{21}^{(1)} \qquad \imath(f_0) = t_{12}^{(1)} \qquad \imath(\xi_0) = t_{22}^{(1)} - t_{11}^{(1)}$$

$$\imath(\xi_1) = t_{22}^{(2)} - t_{11}^{(2)} + \mathsf{h}\left((t_{11}^{(1)})^2 - t_{22}^{(1)} t_{11}^{(1)} - t_{21}^{(1)} t_{12}^{(1)}\right)$$

which implies that the element $t_1 = \xi_1 - \mathsf{h}\xi_0^2/2$ is mapped to

$$\imath(t_1) = t_{22}^{(2)} - t_{11}^{(2)} + \frac{\mathsf{h}}{2}(t_{11}^{(1)} - t_{22}^{(1)})(I + 1) - \frac{\mathsf{h}}{2}\kappa \tag{3.1}$$

where $I = t_{11}^{(1)} + t_{22}^{(1)}$ and $\kappa = t_{12}^{(1)} t_{21}^{(1)} + t_{21}^{(1)} t_{12}^{(1)}$.

[2] we follow the sign conventions of [**22**].

REMARK. The restriction of $\imath$ to $\mathfrak{sl}_2 \subset Y_{\mathsf{h}}\mathfrak{sl}_2$ is not the standard embedding $\jmath : \mathfrak{sl}_2 \to \mathfrak{gl}_2$ given by $e \to E_{12}, f \to E_{21}, h \to E_{11} - E_{22}$. In fact, $\imath|_{\mathfrak{sl}_2} = \theta \circ \jmath$, where $\theta \in \mathrm{Aut}(\mathfrak{gl}_2)$ is the Chevalley involution given by

$$\theta(E_{ij}) = E_{\overline{i}\,\overline{j}} \tag{3.2}$$

with $\overline{1} = 2$ and $\overline{2} = 1$.

3.3. The trigonometric Casimir connection of $\mathfrak{gl}_2$ [28]. Let $T \subset GL_2$ be the maximal torus consisting of diagonal matrices and $\mathfrak{t}$ its Lie algebra. The trigonometric Casimir connection of $\mathfrak{gl}_2$ is the $Y_{\mathsf{h}}\mathfrak{gl}_2$–valued connection on T given by

$$\widehat{\nabla}_C^{\mathfrak{gl}_2} = d - \mathsf{h}\frac{d(\varepsilon_1 - \varepsilon_2)}{e^{\varepsilon_1 - \varepsilon_2} - 1}\kappa - d\varepsilon_1 \mathcal{A}_1 - d\varepsilon_2 \mathcal{A}_2 \tag{3.3}$$

where

(1) $\{\varepsilon_1, \varepsilon_2\}$ is the basis of $\mathfrak{t}^*$ given by $\varepsilon_i(E_{jj}) = \delta_{ij}$ and $\{d\varepsilon_i\}$ are the corresponding translation–invariant 1–forms on T.
(2) The elements $\mathcal{A}_1, \mathcal{A}_2 \in Y_{\mathsf{h}}\mathfrak{gl}_2$ are given by

$$\mathcal{A}_1 = 2t_{11}^{(2)} - \mathsf{h}(t_{11}^{(1)})^2 - \mathsf{h}t_{11}^{(1)}$$
$$\mathcal{A}_2 = 2t_{22}^{(2)} - \mathsf{h}(t_{22}^{(1)})^2 - \mathsf{h}t_{22}^{(1)} - \mathsf{h}\kappa$$

Let the symmetric group $\mathfrak{S}_2$ act on $Y_{\mathsf{h}}\mathfrak{gl}_2$ by $\sigma(t_{ij}^{(r)}) = t_{\sigma(i),\sigma(j)}^{(r)}$, and regard $Y_{\mathsf{h}}\mathfrak{sl}_2$ as embedded in $Y_{\mathsf{h}}\mathfrak{gl}_2$ via 3.2.

THEOREM. [**28**, §5]

(1) *The trigonometric Casimir connection $\widehat{\nabla}_C^{\mathfrak{gl}_2}$ is a flat, $\mathfrak{S}_2$–equivariant connection on the trivial vector bundle $T \times Y_{\mathsf{h}}\mathfrak{gl}_2$.*
(2) *The restriction of $\widehat{\nabla}_C^{\mathfrak{gl}_2}$ to the maximal torus H of SL_2 is the trigonometric Casimir connection $\widehat{\nabla}_C^{\mathfrak{sl}_2}$ of $\mathfrak{sl}_2$.*

3.4. Evaluation homomorphism. The Yangian $Y_{\mathsf{h}}\mathfrak{gl}_2$ admits a one–parameter family of algebra homomorphisms $\mathrm{ev}_a : Y_{\mathsf{h}}\mathfrak{gl}_2 \to U\mathfrak{gl}_2[\mathsf{h}]$ labelled by $a \in \mathbb{C}[\mathsf{h}]$, and given by

$$\mathrm{ev}_a(t_{ij}^{(r)}) = a^{r-1}E_{ij}$$

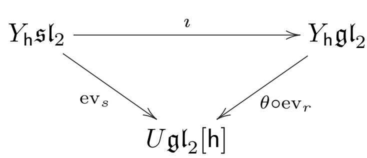
Note that this expression continues to make sense, and to define a homomorphism $Y_{\mathsf{h}}\mathfrak{gl}_2 \to U\mathfrak{gl}_2[\mathsf{h}]$ if a is a central element in $U\mathfrak{gl}_2[\mathsf{h}]$.

The evaluation homomorphism of $Y_\hbar(\mathfrak{gl}_2)$ does not restrict to the one for $Y_{\mathsf{h}}\mathfrak{sl}_2$ defined in 2.3. However, the following holds

LEMMA. *If the evaluation points are related by*

$$r = s + \frac{\mathsf{h}}{2}(I + 1) \tag{3.4}$$

the following diagram is commutative



where θ is the Chevalley involution 3.2.

PROOF. This clearly holds for the generators e_0, f_0, ξ_0 of $Y_\mathsf{h}\mathfrak{sl}_2$, and follows for the element t_1 by comparing

$$\mathrm{ev}_r(\imath(t_1)) = (E_{11} - E_{22})(-r + \frac{\mathsf{h}}{2}(I+1)) - \frac{\mathsf{h}}{2}\kappa$$

where we used (3.1), with $\mathrm{ev}_s(t_1) = sh - \mathsf{h}\kappa/2$. □

3.5. Now let $(s_1, \ldots, s_k) \in \mathbb{C}[\mathsf{h}]^k$ and set $r_a = s_a + \frac{\mathsf{h}}{2}(I+1)$ for any $1 \leq a \leq k$, as in (3.4). Consider the algebra homomorphism

$$\mathrm{ev}_{\underline{r}} = \mathrm{ev}_{r_1} \otimes \cdots \otimes \mathrm{ev}_{r_k} \circ \Delta^{(k)} : Y_\mathsf{h}\mathfrak{gl}_2 \to U\mathfrak{gl}_2^{\otimes k}[h]$$

PROPOSITION. [**28**, Prop. 5.6]

(1) *The image of* $\widehat{\nabla}_C^{\mathfrak{gl}_2}$ *under the evaluation homomorphism* $\mathrm{ev}_{\underline{r}}$ *is the* $U\mathfrak{gl}_2^{\otimes k}[\mathsf{h}]$*–valued connection given by*

$$\widehat{\nabla}_{C,\underline{s}}^{\mathfrak{gl}_2} = d - \mathsf{h}\frac{d(\varepsilon_1 - \varepsilon_2)}{e^{\varepsilon_1 - \varepsilon_2} - 1}\Delta^{(k)}(\kappa) - d\varepsilon_1 A_1 - \varepsilon_2 A_2 \tag{3.5}$$

where

$$A_1 = \sum_a (2s_a E_{11} + \mathsf{h}E_{11}E_{22})^{(a)} + 2\mathsf{h}\sum_{a<b} E_{12}^{(a)} E_{21}^{(b)}$$
$$A_2 = \sum_a (2s_a E_{22} + \mathsf{h}E_{11}E_{22})^{(a)} - 2\mathsf{h}\sum_{a<b} E_{12}^{(a)} E_{21}^{(b)} - \mathsf{h}\sum_a \kappa^{(a)}$$

(2) *The restriction of* $\widehat{\nabla}_{C,\underline{s}}^{\mathfrak{gl}_2}$ *to* $H \subset T$ *is the image of the* $U\mathfrak{sl}_2^{\otimes k}[\mathsf{h}]$*–valued connection* $\widehat{\nabla}_{C,\underline{s}}^{\mathfrak{sl}_2}$ *of Proposition 2.4 under the Chevalley involution* $\theta^{\otimes k}$.

PROOF. (1) By 3.1, $\Delta(t_{ii}^{(2)}) = t_{ii}^{(2)} \otimes 1 + 1 \otimes t_{ii}^{(2)} + \mathsf{h}\sum_{i'} t_{ii'}^{(1)} \otimes t_{i'i}^{(1)}$, which implies that

$$\Delta^{(k)}(t_{ii}^{(2)}) = \sum_a (t_{ii}^{(2)})^{(a)} + \mathsf{h}\sum_{a<b}(t_{i\bar{i}}^{(1)})^{(a)}(t_{\bar{i}i}^{(1)})^{(b)} + \mathsf{h}\sum_{a<b}(t_{ii}^{(1)})^{(a)}(t_{ii}^{(1)})^{(b)}$$

where $\overline{1} = 2, \overline{2} = 1$. Since $\Delta^{(k)}(t_{ii}^{(1)})^2 = 2\sum_{a<b}(t_{ii}^{(1)})^{(a)}(t_{ii}^{(1)})^{(b)} + \sum_a((t_{ii}^{(1)})^{(a)})^2$, this yields

$$\mathrm{ev}_{\underline{r}}\left(2t_{ii}^{(2)} - \mathsf{h}(t_{ii}^{(1)})^2 - \mathsf{h}t_{ii}^{(1)}\right) = \sum_a (E_{ii}(2r_a - \mathsf{h}(E_{ii}+1)))^{(a)} + 2\mathsf{h}\sum_{a<b}(t_{i\bar{i}}^{(1)})^{(a)}(t_{\bar{i}i}^{(1)})^{(b)}$$

Substituting $r_a = s_a + \frac{\mathsf{h}}{2}(I+1)$ yields the claimed formula for $A_1 = \mathrm{ev}_{\underline{r}}(\mathcal{A}_1)$. The formula for A_2 follows from the above, and the fact that

$$\Delta^{(k)}(\kappa) = \sum_a \kappa^{(a)} + 2\sum_{a<b}\left((t_{12}^{(1)})^{(a)}(t_{21}^{(1)})^{(b)} + (t_{21}^{(1)})^{(a)}(t_{12}^{(1)})^{(b)}\right)$$

(2) is a direct consequence of Proposition 3.3 and Lemma 3.4. □

REMARKS.

(1) Since the Chevalley involution is given by conjugating by the matrix $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in SL_2$, the application of $\theta^{\otimes k}$ to the connection $\widehat{\nabla}_{C,\underline{s}}^{\mathfrak{sl}_2}$ yields a connection with the same monodromy.

(2) As shown in [**28**, §5.15], the connection $\widehat{\nabla}^{\mathfrak{gl}_2}_{C,\underline{k}}$ coincides, modulo abelian terms, with the trigonometric dynamical differential equations for $\mathfrak{gl}_2$ considered in [**25**].

4. Affine braid groups

4.1. Set

$$B_{SL_2} = \pi_1(H_{\text{reg}}/W) \qquad \text{and} \qquad B_{GL_2} = \pi_1(T_{\text{reg}}/W)$$

The following is well known [**7, 12, 21**]

PROPOSITION.

(1) *B_{SL_2} is the affine braid group of type A_1, and hence admits the presentation*

$$B_{SL_2} = \langle S_0, S_1 | \text{ no relations } \rangle$$

(2) *B_{GL_2} can be realised as the subgroup of the Artin braid group on three strands B_3, consisting of braids where the first strand is fixed. It has the presentation*

$$B_{GL_2} = \langle \mathcal{X}_1, b | \, b\mathcal{X}_1 b\mathcal{X}_1 = \mathcal{X}_1 b\mathcal{X}_1 b\rangle$$

4.2. We describe the generators $S_0, S_1, b, \mathcal{X}_1$ below, together with the inclusion $B_{SL_2} \subset B_{GL_2}$ stemming from the W–equivariant embedding $H_{\text{reg}} \subset T_{\text{reg}}$.

Identify to this end the tori H and T with $\mathbb{C}^\times$ and $(\mathbb{C}^\times)^2$ respectively, by

$$z \to \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \qquad \text{and} \qquad (z_1, z_2) \to \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$$

In terms of these identifications, the inclusion $H \subset T$ is given by $z \mapsto (z, z^{-1})$. Moreover, $H_{\text{reg}} \subset H$ and $T_{\text{reg}} \subset T$ are identified with $\mathbb{C}^\times \setminus \{\pm 1\}$ and $Y_2(\mathbb{C}^\times)$ respectively, where the latter is the configuration space of two ordered points in $\mathbb{C}^\times$.

4.3. The generators S_0, S_1 of B_{SL_2} may be described as follows [**23, 29, 30**]. Identify the Lie algebra $\mathfrak{h}$ of H with $\mathbb{C}$ by mapping h to 1. The exponential map $\exp(2\pi\iota -) : \mathfrak{h} \to H$ maps $\mathfrak{h}^{\text{a-reg}}$ to H_{reg}, where

$$\mathfrak{h}^{\text{a-reg}} = \mathfrak{h} \setminus \bigcup_{n \in \mathbb{Z}} \{\alpha = n\} \cong \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$$

The affine Weyl group W_{aff} of type A_1 is generated by the affine (real) reflections s_0, s_1 through the points $u = 1/2$ and $u = 0$ respectively. W_{aff} is isomorphic to $\mathbb{Z}_2 \ltimes \mathbb{Z}$, with the generator s_1 of $\mathbb{Z}_2$ acting on $\mathfrak{h}$ as the reflection $u \to -u$ and the generator $\tau = s_0 s_1$ of $\mathbb{Z}$ as the translation $u \to u + 1$. Thus we have the identification

$$\exp(2\pi\iota -) : \mathfrak{h}^{\text{a-reg}}/W_{\text{aff}} \cong H_{\text{reg}}/W \tag{4.1}$$

Fix now a base point (say $u = 1/4$) in $\mathfrak{h}^{\text{a-reg}}$ lying in the interval $(0, 1/2)$. Then, the generators S_i are represented by the loops in $\mathfrak{h}^{\text{a-reg}}/W_{\text{aff}}$ given in Figure 4.1. These correspond, via the identification (4.1) to the loops in H_{reg} shown in Figure 4.2.

4.4. Turning to the fundamental group B_{GL_2}, it is conventional to pick its base point as the configuration $(1, 2)$ in $\mathbb{C}^\times$. The generators $\mathcal{X}_1, b$ are then represented by the braids in Figure 4.3. Set $\mathcal{X}_2 = b\mathcal{X}_1 b$. Then, the defining relation of B_{GL_2} can be written as $\mathcal{X}_1\mathcal{X}_2 = \mathcal{X}_2\mathcal{X}_1$.

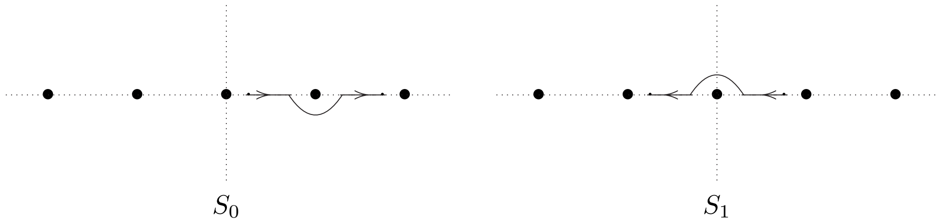


FIGURE 4.1. Generators of $\pi_1\left(\mathfrak{h}^{\text{a-reg}}/W_{\text{aff}}\right)$

FIGURE 4.2. Generators of $\pi_1(H_{\text{reg}}/W)$

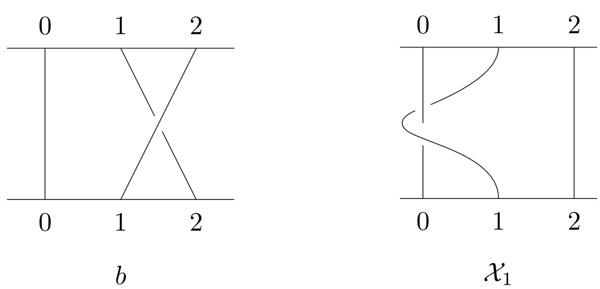


FIGURE 4.3. Generators of B_{GL_2}

4.5. To relate B_{SL_2} and B_{GL_2}, think of elements of B_{SL_2} as braids with 3 strands, with endpoints $-i, 0, i$, and the strand at zero remaining fixed. Choosing a path from the base point $(-i, 0, i)$ to $(0, 1, 2)$ which first braids the first two points to $(0, i/2, i)$ while keeping the third fixed and then scales the configuration to $(0, 1, 2)$ yields an embedding $B_{SL_2} \to B_{GL_2}$ given by

$$\begin{array}{rcl} S_1 & \mapsto & b \\ S_0 & \mapsto & \mathcal{X}_1 b \mathcal{X}_1^{-1} \end{array} \tag{4.2}$$

Let $\mathcal{L} = S_0 S_1$ be the element of B_{SL_2} corresponding to the generator of the coroot lattice of SL_2. Then, the inclusion above yields

$$\mathcal{L} \mapsto \mathcal{X}_1 b \mathcal{X}_1^{-1} b = b^{-1}(b\mathcal{X}_1 b)\mathcal{X}_1^{-1} b = b^{-1}(\mathcal{X}_2 \mathcal{X}^{-1})b$$

Thus, if we consider the set $\{b, \mathcal{L}_1, \mathcal{L}_2\}$ of generators of B_{GL_2} obtained by conjugation with b^{-1}

$$\mathcal{L}_1 = b^{-1}\mathcal{X}_2 b \qquad \text{and} \qquad \mathcal{L}_2 = b^{-1}\mathcal{X}_1 b$$

then, the image of the element $\mathcal{L}$ of B_{SL_2} is given by

$$\mathcal{L} \mapsto \mathcal{L}_1 \mathcal{L}_2^{-1} \tag{4.3}$$

4.6. We now relate the monodromy representations of the trigonometric Casimir connections $\widehat{\nabla}^{\mathfrak{sl}_2}_{C,\underline{s}}$ and $\widehat{\nabla}^{\mathfrak{gl}_2}_{C,\underline{s}}$. Note first that the actions of $W \cong \mathbb{Z}/2\mathbb{Z}$ on the fibers of the corresponding vector bundles are different. This difference arises from the fact that in the case of $SL_2(\mathbb{C})$, the following element is used to construct a group homomorphism $\widetilde{W} \to SL_2(\mathbb{C})$ (see the discussion preceding [**26**, Corollary 3.6]).

$$\sigma \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let $\pi^{\mathfrak{sl}_2}_{C,\underline{s}}$ and $\pi^{\mathfrak{gl}_2}_{C,\underline{s}}$ denote the representations of B_{SL_2} and B_{GL_2} obtained from the monodromy of the connections $\widehat{\nabla}^{\mathfrak{sl}_2}_{C,\underline{s}}$ and $\widehat{\nabla}^{\mathfrak{gl}_2}_{C,\underline{s}}$ respectively. Then we have

$$\begin{aligned} \pi^{\mathfrak{sl}_2}_{C,\underline{s}}(b) &= \pi^{\mathfrak{gl}_2}_{C,\underline{s}}(b)(-1)^{E_{11}} \\ \pi^{\mathfrak{sl}_2}_{C,\underline{s}}(\mathcal{L}) &= \pi^{\mathfrak{gl}_2}_{C,\underline{s}}(\mathcal{L}_2\mathcal{L}_1^{-1}) \end{aligned} \tag{4.4}$$

5. The trigonometric KZ equations

5.1. Fix $k \geq 2$, let $r \in \mathfrak{gl}_k^{\otimes 2}$ be the Drinfeld r–matrix of $\mathfrak{gl}_k$,

$$r = \frac{1}{2}\sum_{a=1}^{k} E_{aa} \otimes E_{aa} + \sum_{1 \leq a < b \leq k} E_{ab} \otimes E_{ba}$$

and let $r(u) = \dfrac{re^u + r_{21}}{e^u - 1}$ be the corresponding trigonometric r–matrix. Fix $n \geq 1$, let V be a $\mathfrak{gl}_k$–module and $\mathbb{V}^{\otimes n}$ the trivial vector bundle over $\mathbb{C}^n$ with fibre $V^{\otimes n}$. The symmetric group $\mathfrak{S}_n$ acts both on the base and fibre of $\mathbb{V}^{\otimes n}$. The trigonometric KZ connection is the flat, $\mathfrak{S}_n$–equivariant connection on $\mathbb{V}^{\otimes n}$ given by

$$\nabla_{KZ} = d - 2\mathsf{h}\left(\sum_{i<j} r_{ij}(u_i - u_j)d(u_i - u_j) + \sum_i s^{(i)}\, du_i\right)$$

where $s \in \mathfrak{gl}_k$ is a fixed diagonal matrix and $s^{(i)} = 1^{\otimes(i-1)} \otimes s \otimes 1^{\otimes(n-i)}$. Since $r(u) = \Omega/(e^u - 1) + r$, where $\Omega = r + r_{21}$, this connection may equivalently be written as

$$\nabla_{KZ} = d - 2\mathsf{h}\left(\sum_{i<j} \frac{d(u_i - u_j)}{e^{u_i - u_j} - 1}\,\Omega_{ij} + \sum_i du_i\, X_i\right) \tag{5.1}$$

where $X_i = s^{(i)} + \sum_{j>i} r_{ij} - \sum_{j<i} r_{ji}$.

5.2. The connection ∇_{KZ} is invariant under the group $\mathbb{Z}^n$ acting trivially on the fibres of $\mathbb{V}^{\otimes n}$ and by translations by the lattice $2\pi\imath\mathbb{Z}^n$ on the base. It therefore descends to a flat connection on the complement $\mathfrak{X}_n$, in the quotient $\mathbb{C}^n/\mathfrak{S}_n \ltimes \mathbb{Z}^n$ of the images of the affine hyperplanes $\{u_i - u_j = 2\pi\imath m\}_{i\neq j, m\in\mathbb{Z}}$. The latter may be thought of as the configuration space of n points in $\mathbb{C}^\times$, or equivalently the set of regular elements in the maximal torus of diagonal matrices in $GL_n(\mathbb{C})$. The following gives a presentation of the fundamental group $\Pi_n = B_{GL_n}$ of this space

Proposition. [**3**] *Π_n is generated by elements $\{b_i\}_{1\leq i\leq n-1}$ and $\{X_j\}_{1\leq j\leq n}$, subject to the relations*

$$\begin{aligned} b_i b_{i'} &= b_{i'} b_i \\ b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1} \\ b_i \mathcal{X}_i b_i &= \mathcal{X}_{i+1} \\ \mathcal{X}_j \mathcal{X}_k &= \mathcal{X}_k \mathcal{X}_j \end{aligned}$$

for any $1 \leq i, i' \leq n-1$ such that $|i - i'| \geq 2$, and $1 \leq j, k \leq n$.

The generators $b_i, \mathcal{X}_j$ may be described as follows. Let $z_i = e^{u_i}$, $i = 1, \ldots, n$, be the standard coordinates on $\mathfrak{X}_n$ and, for definiteness, choose $\underline{z}_0 = (1, \ldots, n)$ as basepoint. Then, $\mathcal{X}_j$ and b_i are, respectively, the loops

$$\begin{aligned} &t \mapsto (1, \ldots, j-1, e^{2\pi\imath t} j, j+1, \ldots, n) \\ &t \mapsto (1, \ldots, i-1, i + 1/2(1 - e^{\pi\imath t}), i + 1/2(1 + e^{\pi\imath t}), i+2, \ldots, n) \end{aligned}$$

where $t \in [0, 1]$.

6. The dual pair $(\mathfrak{gl}_k, \mathfrak{gl}_2)$ and trigonometric connections

6.1. Let $\mathcal{M}_{k,2}$ be the vector space of complex, $k \times 2$ matrices and

$$\mathbb{C}[\mathcal{M}_{k,2}] = \mathbb{C}[x_{aj}]_{\substack{1\leq a\leq k \\ 1\leq j\leq 2}}$$

its algebra of regular functions. The group $GL_k \times GL_2$ acts on $\mathbb{C}[\mathcal{M}_{k,2}]$ by

$$(g_k, g_2)p(X) = p(g_k^t\, X\, g_2)$$

where $X \in \mathcal{M}_{k,2}$ and $g_p \in GL_p$. Note that

$$\mathbb{C}[x_{a1}]\otimes\mathbb{C}[x_{a2}] \cong \mathbb{C}[\mathcal{M}_{k,1}]^{\otimes 2} \cong \mathbb{C}[\mathcal{M}_{k,2}] \cong \mathbb{C}[\mathcal{M}_{1,2}]^{\otimes k} \cong \mathbb{C}[x_{1j}]\otimes\cdots\otimes\mathbb{C}[x_{kj}] \quad (6.1)$$

where the first two are isomorphisms of GL_k–modules, and the last two of GL_2–modules.

The action of $GL_k \times GL_2$ preserves the finite–dimensional homogeneous components of $\mathbb{C}[\mathcal{M}_{k,2}]$ and therefore gives rise to an action of $\mathfrak{gl}_k \oplus \mathfrak{gl}_2$ on these. To distinguish between the elements of these Lie algebras, we denote by $X^{(p)}$ the elements of $\mathfrak{gl}_p$. Then, the action is given by mapping the elementary matrices $E^{(k)}_{ab}, E^{(2)}_{ij}$ to

$$E^{(k)}_{ab} \mapsto \sum_{j=1}^{2} x_{aj}\partial_{bj} \qquad\qquad E^{(2)}_{ij} \mapsto \sum_{a=1}^{k} x_{ai}\partial_{aj} \quad (6.2)$$

6.2.

LEMMA. *The following holds on* $\mathbb{C}[\mathcal{M}_{1,2}]$

$$\kappa^{(2)} = I^{(2)} + 2E_{11}^{(2)}E_{22}^{(2)}$$

PROOF. On $\mathbb{C}[\mathcal{M}_{1,2}] \cong \mathbb{C}[x_j]_{j=1,2}$, $\kappa = E_{12}E_{21} + E_{21}E_{12}$ acts as

$$x_1\partial_2 x_2\partial_1 + x_2\partial_1 x_1\partial_2 = x_1\partial_1 x_2\partial_2 + x_1\partial_1 + x_2\partial_2 x_1\partial_1 + x_2\partial_2 = 2E_{11}E_{22} + I$$

□

6.3. Duality. The identities below relate the coefficients of the KZ connection of $\mathfrak{gl}_k$ and those of the Casimir connection of $\mathfrak{gl}_n$ (here, $n = 2$). They were discovered in [**26**]. Let $r = r^{(k)} \in \mathfrak{gl}_k^{\otimes 2}$ be the r–matrix defined in §5.1 and $\Omega^{(k)} = r + r_{21}$.

PROPOSITION. *The following identities hold on* $\mathbb{C}[\mathcal{M}_{k,1}]^{\otimes 2} \cong \mathbb{C}[\mathcal{M}_{k,2}] \cong \mathbb{C}[\mathcal{M}_{1,2}]^{\otimes k}$

$$(E_{aa}^{(k)})^{(i)} = (E_{ii}^{(2)})^{(a)}$$

$$r^{(k)} = \sum_{a<b}(E_{12}^{(2)})^{(a)}(E_{21}^{(2)})^{(b)} + \frac{1}{2}\sum_a (E_{11}^{(2)}E_{22}^{(2)})^{(a)}$$

$$2\Omega^{(k)} = \Delta^{(k)}(\kappa^{(2)} - I^{(2)})$$

$$s^{(i)} = \sum_a s_a (E_{ii}^{(2)})^{(a)}$$

PROOF. (1) The identity follows from the fact that both sides record the homogeneity degree with respect to the variable x_{ai}.

(2) By (6.2), the action of $\overline{r} = r - \frac{1}{2}\sum_a E_{aa}^{(k)} \otimes E_{aa}^{(k)}$ is given by

$$\sum_{a<b} x_{a1}\partial_{b1}x_{b2}\partial_{a2} = \sum_{a<b} x_{a1}\partial_{a2}x_{b2}\partial_{b1} = \sum_{a<b}(E_{12}^{(2)})^{(a)}(E_{21}^{(2)})^{(b)}$$

and, by (1), $r - \overline{r}$ acts by $\frac{1}{2}\sum_a (E_{11}^{(2)}E_{22}^{(2)})^{(a)}$.

(3) By (2),

$$\begin{aligned}\Omega^{(k)} &= r + r^{21}\\ &= \sum_{a<b}\left((E_{12}^{(2)})^{(a)}(E_{21}^{(2)})^{(b)} + (E_{21}^{(2)})^{(a)}(E_{12}^{(2)})^{(b)}\right) + \sum_a (E_{11}^{(2)}E_{22}^{(2)})^{(a)}\\ &= \sum_{a\neq b}(E_{12}^{(2)})^{(a)}(E_{21}^{(2)})^{(b)} + \frac{1}{2}\sum_a (\kappa^{(2)} - I^{(2)})^{(a)}\end{aligned}$$

where we used Lemma 6.2. On the other hand, since $\kappa^{(2)} = E_{12}^{(2)}E_{21}^{(2)} + E_{21}^{(2)}E_{12}^{(2)}$,

$$\begin{aligned}\Delta^{(k)}(\kappa^{(2)}) &= \sum_{a,b}(E_{12}^{(2)})^{(a)}(E_{21}^{(2)})^{(b)} + (E_{21}^{(2)})^{(b)}(E_{12}^{(2)})^{(a)}\\ &= 2\sum_{a\neq b}(E_{12}^{(2)})^{(a)}(E_{21}^{(2)})^{(b)} + \sum_a (\kappa^{(2)})^{(a)}\end{aligned}$$

(4) follows from (1) since

$$s^{(i)} = \sum_a s_a (E^{(k)}_{aa})^{(i)} = \sum_a s_a (E^{(2)}_{ii})^{(a)}$$

□

6.4. The following is a direct consequence of Proposition 6.3 (see also [**25**]).

PROPOSITION. *Under the identification*

$$\mathbb{C}[\mathcal{M}_{k,1}]^{\otimes 2} \cong \mathbb{C}[\mathcal{M}_{k,2}] \cong \mathbb{C}[\mathcal{M}_{1,2}]^{\otimes k}$$

the trigonometric KZ connection for $\mathfrak{gl}_k$ *with values in* $\mathbb{C}[\mathcal{M}_{k,1}]^{\otimes 2}$ *corresponding to a diagonal matrix* $s = \sum_a s_a E^{(k)}_{aa}$*, coincides with the sum of*

(1) *the trigonometric Casimir connection for* $\mathfrak{gl}_2$ *with values in the tensor product of evaluation modules*

$$\mathbb{C}[\mathcal{M}_{1,2}](r_1) \otimes \cdots \otimes \mathbb{C}[\mathcal{M}_{1,2}](r_k)$$

where $r_a = \mathsf{h}\left(s_a + \frac{I^{(2)}+1}{2}\right)$*, and*

(2) *the closed one–form with values in* $Z(\mathfrak{gl}_2)^{\otimes k}$ *given by*

$$\mathcal{A} = \mathsf{h}\left(\frac{d(\epsilon_1 - \epsilon_2)}{e^{\epsilon_1 - \epsilon_2} - 1} - d\epsilon_2\right) \Delta^{(k)}(I^{(2)})$$

PROOF. Using the form (5.1), it follows from Proposition 6.3 that the trigonometric KZ connection for $\mathfrak{gl}_k$ may be rewritten as the $U\mathfrak{gl}_2^{\otimes k}[\mathsf{h}]$–valued connection

$$d - \mathsf{h}\left(\frac{d(\epsilon_1 - \epsilon_2)}{e^{\epsilon_1 - \epsilon_2} - 1} \Delta^{(k)}(\kappa^{(2)}) + d\epsilon_1 X_1 + d\epsilon_2 X_2\right) + \mathsf{h}\frac{d(\epsilon_1 - \epsilon_2)}{e^{\epsilon_1 - \epsilon_2} - 1} \Delta^{(k)}(I^{(2)})$$

where

$$\begin{aligned} X_1 = 2(s^{(1)} + r) &= \sum_a (2s_a E_{11} + E^{(2)}_{11} E^{(2)}_{22})^{(a)} + 2\sum_{a<b} (E^{(2)}_{12})^{(a)} (E^{(2)}_{21})^{(b)} \\ X_2 = 2(s^{(2)} - r) &= \sum_a (2s_a E^{(2)}_{11} - E^{(2)}_{11} E^{(2)}_{22})^{(a)} - 2\sum_{a<b} (E^{(2)}_{12})^{(a)} (E^{(2)}_{21})^{(b)} \\ &= \sum_a (2s_a E^{(2)}_{11} + E^{(2)}_{11} E^{(2)}_{22})^{(a)} - 2\sum_{a<b} (E^{(2)}_{12})^{(a)} (E^{(2)}_{21})^{(b)} - \sum_a (\kappa^{(2)} - I^{(2)})^{(a)} \end{aligned}$$

where we used Lemma 6.2. The result now follows from Proposition 3.5. □

6.5.

COROLLARY. *Let*

$$\pi_{\mathrm{KZ},\underline{s}}, \pi_{C,s} : B_{GL_2} \to GL\left(\mathbb{C}[\mathcal{M}_{k,2}][[\mathsf{h}]]\right)$$

be the monodromy representations of the trigonometric KZ connection for $\mathfrak{gl}_k$ *and Casimir connection for* $\mathfrak{gl}_2$ *corresponding to the diagonal matrix* $s = \sum_a s_a E^{(k)}_{aa}$ *and evaluation points* $r_a = \mathsf{h}(s_a + \frac{I^{(2)}+1}{2})$ *respectively. Then,*

$$\begin{aligned} \pi_{\mathrm{KZ},\underline{s}}(b) &= \pi_{C,\underline{s}}(b) e^{-\pi\iota\mathsf{h}\Delta^{(k)}(I^{(2)})} \\ \pi_{\mathrm{KZ},\underline{s}}(\mathcal{X}_1) &= \pi_{C,\underline{s}}(\mathcal{X}_1) e^{2\pi\iota\mathsf{h}\Delta^{(k)}(I^{(2)})} \\ \pi_{\mathrm{KZ},\underline{s}}(\mathcal{X}_2) &= \pi_{C,\underline{s}}(\mathcal{X}_2) \end{aligned}$$

PROOF. In terms of the coordinates $z_1 = e^{\varepsilon_1}$, $z_2 = e^{\varepsilon_2}$, the 1–form $\mathcal{A}$ of Proposition 6.4 is equal to

$$\mathsf{h}\left(\frac{d(z_1 - z_2)}{z_1 - z_2} - \frac{dz_1}{z_1} - \frac{dz_2}{z_2}\right)\Delta^{(k)}(I^{(2)})$$

A fundamental solution of the corresponding connection is given by

$$(z_1 z_2/(z_2 - z_1))^{\mathsf{h}\Delta^{(k)}(I^{(2)})} = \exp\left(\mathsf{h}\Delta^{(k)}(I^{(2)})(\log z_1 + \log z_2 - \log(z_2 - z_1))\right)$$

where log is the standard determination of the logarithm, and has monodromy along the generators $b, \mathcal{X}_1, \mathcal{X}_2$ described in Section 5.2 given by

$$b \mapsto e^{-\pi\iota\mathsf{h}\Delta^{(k)}(I^{(2)})} \qquad \mathcal{X}_1 \mapsto e^{2\pi\iota\mathsf{h}\Delta^{(k)}(I^{(2)})} \qquad \mathcal{X}_2 \mapsto 1$$

□

7. Monodromy of the trigonometric KZ equations

In this section we recall the main theorem of [**12**] which computes the monodromy of the trigonometric KZ equations.

7.1. The quantum group $U_\hbar\mathfrak{gl}_p$. The Drinfeld–Jimbo quantum group $U_\hbar\mathfrak{gl}_p$ is defined as a unital associative $\mathbb{C}[[\hbar]]$–algebra, topologically generated by elements $\{E_j, F_j\}_{1\leq j\leq p-1}$ and $\{D_i\}_{1\leq i\leq p}$ subject to the relations (where $q^2 = e^\hbar$)

(QG1) $[D_i, D_{i'}] = 0$ for any i, i'.

(QG2) For each i, j, $1 \leq i \leq p$ and $1 \leq j \leq p-1$ we have

$$[D_i, E_j] = (\delta_{ij} - \delta_{i,j+1})\, E_j \qquad\qquad [D_i, F_j] = (\delta_{i,j+1} - \delta_{ij})\, F_j$$

(QG3) For each $j, j' \in \{1, \cdots, p-1\}$ we have

$$[E_j, F_{j'}] = \delta_{j,j'}\frac{q^{H_j} - q^{-H_j}}{q - q^{-1}}$$

(QG4) For each $j \neq j' \in \{1, \cdots, p-1\}$ we have:

$$\sum_{t=0}^{1-a_{jj'}} (-1)^t \begin{bmatrix} 1 - a_{jj'} \\ t \end{bmatrix}_q E_i^{1-a_{jj'}-t} E_j E_i^t = 0$$

$$\sum_{t=0}^{1-a_{jj'}} (-1)^t \begin{bmatrix} 1 - a_{jj'} \\ t \end{bmatrix}_q F_i^{1-a_{jj'}-t} F_j F_i^t = 0$$

where $H_i = D_i - D_{i+1}$. We have used the standard notations of the Gaussian integers.

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad \text{and} \qquad [n]_q! = [n]_q[n-1]_q \cdots [1]_q$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}$$

and $a_{jj'}$ are entries of the Cartan matrix of type A_{p-1}:

$$a_{jj'} = 2 - \delta_{[j-j'|=1}$$

$U_\hbar\mathfrak{gl}_p$ is a topological Hopf algebra with coproduct and counit given by:

$$\begin{array}{rcl}\Delta(D_i) & = & D_i\otimes 1+1\otimes D_i\\ \Delta(E_j) & = & E_j\otimes q^{H_j}+1\otimes E_j\\ \Delta(F_j) & = & F_j\otimes 1+q^{-H_j}\otimes F_j\end{array} \tag{7.1}$$

and

$$\varepsilon(E_j)=\varepsilon(F_j)=\varepsilon(D_i)=0 \tag{7.2}$$

Let $I_p=D_1+\cdots+D_p\in U_\hbar\mathfrak{gl}_p$. It is clear from the definition above that I_p is a central element of $U_\hbar\mathfrak{gl}_p$, and the coproduct on I_p is given by $\Delta(I_p)=I_p\otimes 1+1\otimes I_p$. We have the following isomorphism of Hopf algebras:

$$U_\hbar\mathfrak{gl}_p=U_\hbar\mathfrak{sl}_p\otimes\mathbb{C}[I_p][[\hbar]]$$

Moreover $U_\hbar\mathfrak{gl}_p$ has a quasitriangular structure. Let $\mathcal{R}$ be the R–matrix of $U_\hbar\mathfrak{gl}_p$. Recall that the Drinfeld element u is defined by:

$$u=(m\circ(S\otimes 1))(\mathcal{R}_{21}) \tag{7.3}$$

The following theorem is proved in [**11**].

THEOREM. *The square of the antipode is an inner automorphism given by:*

$$S^2(x)=uxu^{-1}$$

REMARK. In this note $\mathcal{R}$ denotes the R–matrix of $U_\hbar\mathfrak{gl}_p$, which differs from the R–matrix of $U_\hbar\mathfrak{sl}_p$ (the one used in [**26**]) by:

$$\mathcal{R}_{\mathfrak{gl}_p}=q^{\frac{I_p\otimes I_p}{p}}\mathcal{R}_{\mathfrak{sl}_p} \tag{7.4}$$

7.2. Monodromy of the trigonometric KZ equations [12]. Let V be a $\mathfrak{gl}_k$–module on which I acts semisimply, $n\geq 1$ and consider the monodromy of the trigonometric KZ equations defined in Section 5 on $V^{\otimes n}$.

Let $\mathcal{V}$ be a finite–dimensional $U_\hbar\mathfrak{gl}_k$–module satisfying $\mathcal{V}/\hbar\mathcal{V}\cong V$ and such that I acts semisimply and with eigenvalues in $\mathbb{C}$. Define

$$T=(S\otimes\mathrm{id})(\mathcal{R}_{21})\qquad\text{and}\qquad C=m_{01}\left(T_{0n}\cdots T_{01}\right)=m_{01}\left((1\otimes\Delta^{(n)})T\right) \tag{7.5}$$

The following is the main result of [**12**]. It relies upon the fact that the Etingof–Kazhdan quantization of $\mathfrak{gl}_k$ corresponding to the r–matrix given in 5.1 coincides with the Drinfeld–Jimbo quantum group $U_\hbar\mathfrak{gl}_k$, which is proved in [**14**].

THEOREM. *Let $\hbar=4\pi\iota h$. Then, the monodromy representation $\pi:\Pi_n\to GL\left(V^{\otimes n}[[\hbar]]\right)$ corresponding to* (5.1) *is equivalent to the following representation of Π_n on $\mathcal{V}^{\otimes n}$*

$$b_i\mapsto(i\ i+1)\mathcal{R}_{i,i+1}$$
$$\mathcal{X}_1\mapsto(q^{2s}u^{-1})^{(1)}\,C$$

The statement of the above theorem differs slightly from the one given in [**12**], due to minor computational errors in [**12**]. For reader's convenience, we reproduce the proof of this theorem in Appendix A.

7.3.

COROLLARY. *The monodromy of the trigonometric KZ connection* (5.1) *is equivalent to the action of* Π_n *on* $\mathcal{V}^{\otimes n}$ *given by*

$$\begin{aligned}\rho_s(b_i) &= (i\,i+1)\mathcal{R}_{i\,i+1}\\ \rho_s(\mathcal{X}_j) &= \mathcal{R}_{j\,j-1}\cdots\mathcal{R}_{j\,1}\,(q^{2s})^{(j)}\,\mathcal{R}_{n\,j}^{-1}\cdots\mathcal{R}_{j+1\,j}^{-1}\\ &= \Delta^{(j-1)}\otimes\mathrm{id}(\mathcal{R}_{21})\cdot(q^{2s})^{(j)}\cdot 1^{\otimes(j-1)}\otimes\mathrm{id}\otimes\Delta^{(n-j)}(\mathcal{R}_{21}^{-1})\end{aligned}$$

PROOF. We need only check the assignment for $\mathcal{X}_1,\dots,\mathcal{X}_n$. Write $\mathcal{R}=\alpha_i\otimes\beta^i$, where the sum over i is implicit. By (7.5),

$$\begin{aligned}(u^{-1})^{(1)}C &= (u^{-1})^{(1)}m_{01}(T_{0n}\cdots T_{01})\\ &= u^{-1}S(\beta^{i_n})\cdots S(\beta^{i_1})\alpha_{i_1}\otimes\alpha_{i_2}\otimes\cdots\otimes\alpha_{i_n}\\ &= u^{-1}S(\beta^{i_n})\cdots S(\beta^{i_2})u\otimes\alpha_{i_2}\otimes\cdots\otimes\alpha_{i_n}\\ &= S^{-1}(\beta^{i_2}\cdots\beta^{i_n})\otimes\alpha_{i_2}\otimes\cdots\otimes\alpha_{i_n}\\ &= S^{-1}\otimes\Delta^{(n-1)}(\mathcal{R}_{21})\\ &= \mathrm{id}\otimes\Delta^{(n-1)}(\mathcal{R}_{21}^{-1})\end{aligned}$$

where we used $u=S(\beta^{i_1})\alpha_{i_1}$, $\mathrm{Ad}(u)=S^2$, the cabling identity $\Delta^{(n-1)}\otimes\mathrm{id}(\mathcal{R})=\mathcal{R}_{1\,n}\mathcal{R}_{2\,n}\cdots\mathcal{R}_{n-1\,n}$ which implies that

$$\mathrm{id}\otimes\Delta^{(n-1)}(\mathcal{R}_{21})=(1\,2\cdots n)\circ\Delta^{(n-1)}\otimes\mathrm{id}(\mathcal{R})=\mathcal{R}_{2\,1}\mathcal{R}_{3\,1}\cdots\mathcal{R}_{n\,1}$$

and the fact that $(S^{-1}\otimes 1)(\mathcal{R}_{21})=\mathcal{R}_{21}^{-1}$. Thus,

$$\rho_s(\mathcal{X}_1)=(q^{2s})^{(1)}\;\mathrm{id}\otimes\Delta^{(n-1)}(\mathcal{R}_{21}^{-1})=(q^{2s})^{(1)}\,\mathcal{R}_{n\,1}^{-1}\cdots\mathcal{R}_{2\,1}^{-1}$$

The formula for $\mathcal{X}_j$ follows from this by an easy induction using $\mathcal{X}_{j+1}=b_j\mathcal{X}_jb_j$. □

7.4. We shall mostly be interested in the case $n=2$. In this case, the above formulae read

$$\rho_s(b)=(1\,2)\mathcal{R}\qquad\rho_s(\mathcal{X}_1)=(q^{2s})^{(1)}\mathcal{R}_{21}^{-1}\qquad\rho_s(\mathcal{X}_2)=\mathcal{R}_{21}(q^{2s})^{(2)}$$

which, in terms of the generators $b,\mathcal{L}_1=b^{-1}\mathcal{X}_2b,\mathcal{L}_2=b^{-1}\mathcal{X}_1b$ of B_{GL_2} defined in 4.4, yields

$$\rho_s(b)=(1\,2)\mathcal{R}\qquad\rho_s(\mathcal{L}_1)=(q^{2s})^{(1)}\mathcal{R}\qquad\rho_s(\mathcal{L}_2)=\mathcal{R}^{-1}(q^{2s})^{(2)}$$

8. Quantum loop algebras

In this section we review the definitions of the quantum loop algebras $U_\hbar(L\mathfrak{gl}_2)$ and $U_\hbar(L\mathfrak{sl}_2)$ following [**8**] and [**6**] respectively.

8.1. The quantum loop algebra $U_\hbar(L\mathfrak{sl}_2)$ **[8].** $U_\hbar(L\mathfrak{gl}_2)$ is a unital, associative, complete $\mathbb{C}[[\hbar]]$–algebra topologically generated by elements $\{E_r,F_r,D_{1,r},D_{2,r},\}_{r\in\mathbb{Z}}$. To state its defining relations, consider the formal series

$$E(z)=\sum_{r\in\mathbb{Z}}E_rz^{-r}\qquad F(z)=\sum_{r\in\mathbb{Z}}F_rz^{-r}$$

$$\Theta_j^{\pm}(z)=q^{\pm D_{j,0}}\exp\left(\pm(q-q^{-1})\sum_{r\geq 1}D_{j,\pm r}z^{\mp r}\right)$$

Then the relations can be written as[3]

(QL1) The elements $\{D_{j,r}\}_{j=1,2;r\in\mathbb{Z}}$ commute.

(QL2) Let $\theta_m(\zeta) = \frac{q^m\zeta-1}{\zeta-q^m}$, then

$$\Theta_1^{\pm}(z)E(w)\Theta_1^{\pm}(z)^{-1} = \theta_1(qz/w)E(w) \qquad \Theta_2^{\pm}(z)E(w)\Theta_2^{\pm}(z)^{-1} = \theta_{-1}(q^{-1}z/w)E(w)$$

$$\Theta_1^{\pm}(z)^{-1}F(w)\Theta_1^{\pm}(z) = \theta_1(qz/w)F(w) \qquad \Theta_2^{\pm}(z)^{-1}F(w)\Theta_2^{\pm}(z) = \theta_{-1}(q^{-1}z/w)F(w)$$

(QL3)

$$E(z)E(w) = \theta_2(z/w) \quad E(w)E(z)$$
$$F(z)F(w) = \theta_2(z/w)^{-1}F(w)F(z)$$

(QL4) Let $\delta(\zeta) = \sum_{n\in\mathbb{Z}} \zeta^n$ be the formal delta function, then

$$(q-q^{-1})[E(z),F(w)] = \delta(z/w)\left(\frac{\Theta_1^+(z)}{\Theta_2^+(z)} - \frac{\Theta_1^-(z)}{\Theta_2^-(z)}\right)$$

8.2. Set

$$\psi^{\pm}(z) = \Theta_1^{\pm}(z)/\Theta_2^{\pm}(z) = K^{\pm 1}\exp\left(\pm(q-q^{-1})\sum_{r\geq 1} H_{\pm r}z^{\mp r}\right) \tag{8.1}$$

where $K = q^{H_0}$ and $H_r = D_{1,r} - D_{2,r}$, $r \in \mathbb{Z}$. Then, the relation (QL4) reads

$$[E_k, F_l] = \frac{\psi_{k+l}^+ - \psi_{k+l}^-}{q-q^{-1}} \tag{8.2}$$

where $\psi_{-p}^+ = \psi_p^- = 0$ for every $p > 0$.

8.3.

Lemma. *The relation (QL2) can be equivalently written as follows. For every $r,k \in \mathbb{Z}$, $r \neq 0$, we have*

$$[D_{j,0}, E_k] = (-1)^{j-1}E_k \qquad [D_{j,0}, F_k] = (-1)^j F_k$$
$$[D_{1,r}, E_k] = \quad q^{-r}\frac{[r]}{r}E_{k+r} \qquad [D_{2,r}, E_k] = -q^r\frac{[r]}{r}E_{k+r}$$
$$[D_{1,r}, F_k] = -q^{-r}\frac{[r]}{r}F_{k+r} \qquad [D_{2,r}, F_k] = \quad q^r\frac{[r]}{r}F_{k+r}$$

And hence we have the following commutation relations

$$[H_0, E_k] = 2E_k \qquad [H_0, F_k] = -2F_k$$
$$[H_r, E_k] = \frac{[2r]}{r}E_{r+k} \qquad [H_r, F_k] = -\frac{[2r]}{r}F_{r+k}$$

[3]The presentation above differs from the one given in [**8**] by the interchange $\Theta_1(z)^{\pm} \leftrightarrow \Theta_2(z)^{\pm}$. The present convention makes the formulae for the inclusion $U_\hbar\mathfrak{gl}_2 \hookrightarrow U_\hbar(L\mathfrak{gl}_2)$ and evaluation $U_\hbar(L\mathfrak{gl}_2) \to U_\hbar\mathfrak{gl}_2$ more natural.

8.4. Quantum determinant. It follows from the relations (QL1)–(QL2) that the coefficients of the series

$$\mathrm{qdet}^{\pm}(z) = \Theta_1^{\pm}(q^{-1}z)\Theta_2^{\pm}(qz)$$

belong to the center of $U_\hbar(L\mathfrak{gl}_2)$. The following result in well–known (see [**22**, Thm. 1.8.2] for the analogous assertion for the Yangian $Y(\mathfrak{gl}_2)$).

PROPOSITION. *The coefficients of* $\mathrm{qdet}^{\pm}(z)$ *generate the center* $\mathcal{Z}$ *of* $U_\hbar(L\mathfrak{gl}_2)$. *Moreover, if* $U_\hbar(L\mathfrak{sl}_2) \subset U_\hbar(L\mathfrak{gl}_2)$ *is the subalgebra generated by* $\{E_r, F_r, H_r\}_{r\in\mathbb{Z}}$, *then*

$$U_\hbar(L\mathfrak{gl}_2) \cong \mathcal{Z} \otimes U_\hbar(L\mathfrak{sl}_2)$$

8.5. Hopf algebra structure. The algebra $U_\hbar(L\mathfrak{gl}_2)$ is a Hopf algebra with comultiplication determined by

$$\begin{array}{rcl} \Delta\left(\mathrm{qdet}^{\pm}(z)\right) &=& \mathrm{qdet}^{\pm}(z) \otimes \mathrm{qdet}^{\pm}(z) \\ \Delta(D_{j,0}) &=& D_{j,0} \otimes 1 + 1 \otimes D_{j,0} \\ \Delta(E_0) &=& E_0 \otimes K + 1 \otimes E_0 \\ \Delta(F_0) &=& F_0 \otimes 1 + K^{-1} \otimes F_0 \\ \Delta(E_{-1}) &=& E_{-1} \otimes K^{-1} + 1 \otimes E_{-1} \\ \Delta(F_1) &=& F_1 \otimes 1 + K \otimes F_1 \end{array} \tag{8.3}$$

8.6. Evaluation homomorphism. For any invertible element $\zeta \in \mathbb{C}[[\hbar]]$, there is a surjective algebra homomorphism $\mathrm{ev}_\zeta : U_\hbar(L\mathfrak{gl}_2) \to U_\hbar\mathfrak{gl}_2$ given on $U_\hbar(L\mathfrak{sl}_2)$ by [**6**, §4.1]

$$H_0 \mapsto D_1 - D_2 \qquad E_0 \mapsto E \qquad F_0 \mapsto F$$

$$E_{-1} \mapsto q\zeta^{-1}K^{-1}E \qquad F_1 \mapsto q^{-1}\zeta FK$$

and on the center $\mathcal{Z}$ by

$$\mathrm{qdet}^{\pm}(z) \mapsto q^I\, \frac{z - q^{-I}\zeta}{z - q^I\zeta}$$

where $I = D_1 + D_2 \in U_\hbar\mathfrak{gl}_2$, and the right–hand side is expanded in powers of $z^{\mp 1}$.

If $\mathcal{V}$ is a $U_\hbar\mathfrak{gl}_2$–module, we denote the $U_\hbar(L\mathfrak{gl}_2)$–module $\mathrm{ev}_\zeta^*(\mathcal{V})$ by $\mathcal{V}(\zeta)$.

8.7. Kac–Moody presentation of $U_\hbar(L\mathfrak{sl}_2)$. The quantum loop algebra $U_\hbar(L\mathfrak{sl}_2)$ is usually presented on Kac–Moody generators $\mathcal{H}, \mathcal{E}_i, \mathcal{F}_i$, $i = 0, 1$, satisfying the relations

(KM1) $[\mathcal{H}, \mathcal{E}_i] = (-1)^{i+1}2\mathcal{E}_i$ and $[\mathcal{H}, \mathcal{F}_i] = (-1)^i 2\mathcal{F}_i$

(KM2) For any $i, j \in \{0, 1\}$

$$[\mathcal{E}_i, \mathcal{F}_j] = \delta_{ij}(-1)^{i+1}\,\frac{q^{\mathcal{H}} - q^{-\mathcal{H}}}{q - q^{-1}}$$

(KM3) For any $i \neq j \in \{0, 1\}$

$$\mathcal{E}_i^3\mathcal{E}_j - [3]\mathcal{E}_i^2\mathcal{E}_j\mathcal{E}_i + [3]\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i^2 - \mathcal{E}_j\mathcal{E}_i^3 = 0$$
$$\mathcal{F}_i^3\mathcal{F}_j - [3]\mathcal{F}_i^2\mathcal{F}_j\mathcal{F}_i + [3]\mathcal{F}_i\mathcal{F}_j\mathcal{F}_i^2 - \mathcal{F}_j\mathcal{F}_i^3 = 0$$

The fact that the relations of Section 8.1 give an equivalent presentation to the ones above is stated in [**10**] (see [**2**] for a proof). The relation between the generators is given by[4]

$$\begin{aligned} &\mathcal{H} = D_{1,0} - D_{2,0} \\ \mathcal{E}_1 &= E_0 \qquad\qquad \mathcal{F}_1 = F_0 \\ \mathcal{E}_0 &= K^{-1}F_1 \qquad\qquad \mathcal{F}_0 = E_{-1}K \end{aligned} \tag{8.4}$$

8.8. Diagram automorphism. Let ω be the diagram automorphism of $U_\hbar(L\mathfrak{sl}_2)$ given by $\mathcal{H} \leftrightarrow -\mathcal{H}$, $\mathcal{E}_0 \leftrightarrow \mathcal{E}_1$ and $\mathcal{F}_0 \leftrightarrow \mathcal{F}_1$. In terms of loop generators, (8.4) shows that ω is given by

$$\begin{aligned} \omega(E_0) &= K^{-1}F_1 \qquad\qquad \omega(F_0) = E_{-1}K \\ \omega(E_{-1}) &= F_0K \qquad\qquad \omega(F_1) = K^{-1}E_0 \end{aligned} \tag{8.5}$$

9. Quantum Weyl groups

In this section, we extend the action of the affine braid group B_{SL_2} on the quantum loop algebra $U_\hbar(L\mathfrak{sl}_2)$ to one of B_{GL_2} on $U_\hbar(L\mathfrak{gl}_2)$. We show that this action is given by conjugating by elements $\mathbb{S}, \mathbb{L}_1, \mathbb{L}_2$, where $\mathbb{S}$ is the quantum Weyl group element of $U_\hbar\mathfrak{sl}_2$ and $\mathbb{L}_1, \mathbb{L}_2$ lie in a completion of the commutative subalgebra of $U_\hbar(L\mathfrak{gl}_2)$ generated by the elements $\{D_{j,k}\}$. The element $\mathbb{L} = \mathbb{L}_1\mathbb{L}_2^{-1}$ is equal to the quantum Weyl group element of $U_\hbar(L\mathfrak{sl}_2)$ corresponding to the generator of the coroot lattice, for which we obtain an explicit expression in terms of the commuting generators H_k.

9.1. Braid group action on $U_\hbar(L\mathfrak{sl}_2)$. Following [**20**], consider the automorphisms T_0, T_1 of $U_\hbar(L\mathfrak{sl}_2)$ given in the Kac–Moody presentation by

$$\begin{gathered} T_0(\mathcal{H}) = -\mathcal{H} = T_1(\mathcal{H}) \\ T_i(\mathcal{E}_i) = -\mathcal{F}_i\mathcal{K}_i \qquad\qquad T_i(\mathcal{F}_i) = -\mathcal{K}_i^{-1}\mathcal{E}_i \end{gathered}$$

where $\mathcal{K}_0 = q^{-\mathcal{H}}$, $\mathcal{K}_1 = q^{\mathcal{H}}$ and, for $i \neq j \in \{0,1\}$,

$$T_i(\mathcal{E}_j) = \mathcal{E}_i^{(2)}\mathcal{E}_j - q^{-1}\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i + q^{-2}\mathcal{E}_j\mathcal{E}_i^{(2)} \qquad T_i(\mathcal{F}_j) = \mathcal{F}_j\mathcal{F}_i^{(2)} - q\mathcal{F}_i\mathcal{F}_j\mathcal{F}_i + q^2\mathcal{F}_i^{(2)}\mathcal{F}_j$$

where $X^{(n)} = X^n/[n]!$.

It is clear that the diagram automorphism ω defined in 8.8 satisfies

$$\omega \circ T_0 \circ \omega = T_1 \tag{9.1}$$

9.2. We shall need for later use the following

Lemma. *The action of T_0 on the loop generators is given by*

$$\begin{aligned} T_0(F_1) &= -K^{-1}E_{-1} \qquad\qquad T_0(E_{-1}) = -F_1K \\ T_0(E_0) &= -K^{-1}F_2 \qquad\qquad T_0(F_0) = -E_{-2}K \end{aligned}$$

[4] we follow here the conventions of [**2**].

PROOF. The first set of equations is a direct consequence of 9.1 and (8.4). We only check the first of the remaining two equations since the second one is verified in a similar way. We have

$$T_0(E_0) = \frac{1}{[2]}(K^{-1}F_1K^{-1}F_1E_0 - (1+q^{-2})K^{-1}F_1E_0K^{-1}F_1 + q^{-2}E_0K^{-1}F_1K^{-1}F_1)$$

We now rewrite each term individually.

$$K^{-1}F_1K^{-1}F_1E_0 = q^{-2}K^{-2}F_1^2E_0 = q^{-2}K^{-2}(F_1E_0F_1 - F_1KH_1)$$

where we used $[E_0, F_1] = \Psi_1^+/(q-q^{-1}) = KH_1$. Next,

$$(1+q^{-2})K^{-1}F_1E_0K^{-1}F_1 = (1+q^{-2})K^{-2}F_1E_0F_1$$

Finally,

$$q^{-2}E_0K^{-1}F_1K^{-1}F_1 = K^{-2}E_0F_1F_1 = K^{-2}(F_1E_0F_1 + KH_1F_1)$$

Combining these computations we get

$$T_0(E_0) = \frac{1}{[2]}K^{-1}[H_1, F_1] = -K^{-1}F_2$$

□

9.3. Let $\omega \in \mathrm{Aut}(U_\hbar(L\mathfrak{sl}_2))$ be the diagram automorphism defined in §8.8.

LEMMA. *The action of $T_0\omega$ on $U_\hbar(L\mathfrak{sl}_2)$ is given in the loop generators by*

$$\psi^\pm(z) \mapsto \psi^\pm(z), \qquad E(z) \mapsto -z^{-1}E(z) \qquad \text{and} \qquad F(z) \mapsto -zF(z)$$

PROOF. It suffices to verify the assertion on the generators K, E_0, E_{-1}, F_0, F_1. It is clear that $T_0\omega$ fixes K. Moreover, using §8.8 and Lemma 9.2, we find

$$T_0(\omega(E_0)) = T_0(K^{-1}F_1) = -E_{-1} \qquad T_0(\omega(E_{-1})) = T_0(F_0K) = -E_{-2}$$
$$T_0(\omega(F_0)) = T_0(E_{-1}K) = -F_1 \qquad T_0(\omega(F_1)) = T_0(K^{-1}E_0) = -F_2$$

□

9.4. The lattice element L. Let $L = T_0T_1 \in \mathrm{Aut}(U_\hbar(L\mathfrak{sl}_2))$. Note that, by (9.1)

$$L = T_0T_1 = T_0\omega T_0\omega = (T_0\omega)^2$$

By Lemma 9.3, the action of L on $U_\hbar(L\mathfrak{sl}_2)$ is therefore given by

$$\psi^\pm(z) \mapsto \psi^\pm(z), \qquad E(z) \mapsto z^{-2}E(z) \qquad \text{and} \qquad F(z) \mapsto z^2F(z) \tag{9.2}$$

9.5. The automorphisms $\mathbf{L_1}, \mathbf{L_2}$. Consider the assignments

$$\begin{array}{llll} L_1 : \Theta_j^\pm(z) \to \Theta_j^\pm(z) & E(z) \to z^{-1}E(z) & F(z) \to z\quad F(z) \\ L_2 : \Theta_j^\pm(z) \to \Theta_j^\pm(z) & E(z) \to z\quad E(z) & F(z) \to z^{-1}F(z) \end{array} \tag{9.3}$$

PROPOSITION.

(1) *L_1 and L_2 extend uniquely to algebra automorphisms of $U_\hbar(L\mathfrak{gl}_2)$ satisfying $L_1L_2 = L_2L_1 = \mathbf{1}$.*
(2) *The automorphism $L = T_0T_1$ is equal to $L_1L_2^{-1}$.*
(3) *L_1 and L_2 satisy*

$$T_1L_2T_1 = L_1$$

and therefore give rise to an action of the affine braid group B_{GL_2} on $U_\hbar(L\mathfrak{gl}_2)$ extending that of B_{SL_2} on $U_\hbar(L\mathfrak{sl}_2)$.

PROOF. (1) It is clear from (9.3) that L_1 and L_2 preserve the defining relations (QL1)–(QL4) of $U_\hbar(L\mathfrak{gl}_2)$ and that $L_1L_2 = L_2L_1 = \mathbf{1}$. (2) Follows by comparing (9.3) and (9.2). (3) It readily follows from Lemma 9.2 and (9.3) that $T_0L_2T_0 = L_1$. Since $T_0T_1 = L = L_1L_2^{-1}$, we have

$$L_1 = T_0L_2T_0 = L_1L_2^{-1}T_1^{-1}L_2L_1L_2^{-1}T_1^{-1} = L_1(L_2^{-1}T_1^{-1}L_1T_1^{-1})$$

Simplifying L_1 yields the claimed identity. □

REMARK. Comparing Lemma 9.3 and (9.3) shows that the restriction of L_1 to $U_\hbar(L\mathfrak{sl}_2)$ satisfies

$$L_1 = \mathrm{Ad}((-1)^{\mathcal{H}/2})\,T_0\,\omega$$

9.6. The Quantum Weyl group of $U_\hbar(L\mathfrak{sl}_2$ [20]. The automorphisms T_0, T_1 are almost inner. Specifically, if $\widehat{U_\hbar(L\mathfrak{sl}_2)}$ is the completion of $U_\hbar(L\mathfrak{sl}_2)$ with respect to its finite–dimensional representations, there are elements $\mathbb{S}_0, \mathbb{S}_1$ in $\widehat{U_\hbar(L\mathfrak{sl}_2)}$ such that conjugation by $\mathbb{S}_i$ preserves $U_\hbar(L\mathfrak{sl}_2)$ and is given by the automorphism T_i. The elements $\mathbb{S}_i$ are given by

$$\begin{aligned}\mathbb{S}_0 &= \exp_{q^{-1}}\left(q^{-1}\mathcal{E}_0\ \ q^{\mathcal{H}}\right)\exp_{q^{-1}}\left(-\mathcal{F}_0\right)\exp_{q^{-1}}\left(q\mathcal{E}_0q^{-\mathcal{H}}\right)q^{\frac{\mathcal{H}(\mathcal{H}-1)}{2}}\\ \mathbb{S}_1 &= \exp_{q^{-1}}\left(q^{-1}\mathcal{E}_1q^{-\mathcal{H}}\right)\exp_{q^{-1}}\left(-\mathcal{F}_1\right)\exp_{q^{-1}}\left(q\mathcal{E}_1\ \ q^{\mathcal{H}}\right)q^{\frac{\mathcal{H}(\mathcal{H}+1)}{2}}\end{aligned} \tag{9.4}$$

where the q–exponential is defined by

$$\exp_q(x) = \sum_{n\geq 0} q^{\frac{n(n-1)}{2}}\frac{x^n}{[n]!}$$

9.7. Completions. We will similarly show that the automorphisms L_1, L_2 are almost inner. We begin by defining appropriate completions of $U_\hbar(L\mathfrak{sl}_2)$ and $U_\hbar(L\mathfrak{gl}_2)$.

For $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{gl}_2$, the completion $\widehat{U_\hbar(L\mathfrak{g})}$ defined below is a flat deformation of the completion $U(\widehat{\mathfrak{g}[z, z^{-1}}])$ of the classical loop algebra with respect to the descending chain of ideals $J_n = U((z-1)^n\mathfrak{g}[z, z^{-1}])$, $n \geq 0$ (see [**16**, Prop. 6.3] for $\mathfrak{g} = \mathfrak{sl}_2$). For $\mathfrak{g} = \mathfrak{sl}_2$, J_n is the nth power of J_1 since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and $\widehat{U_\hbar(L\mathfrak{sl}_2)}$ is correspondingly defined as the completion

$$\widehat{U_\hbar(L\mathfrak{sl}_2)} = \varprojlim U_\hbar(L\mathfrak{sl}_2)/\mathcal{J}^n$$

with respect to the kernel $\mathcal{J}$ of the composition

$$U_\hbar(L\mathfrak{sl}_2) \xrightarrow{\hbar\to 0} U(\mathfrak{sl}_2[z, z^{-1}]) \xrightarrow{z\to 1} U\mathfrak{sl}_2$$

For $\mathfrak{g} = \mathfrak{gl}_2$, the powers of the ideal J_1 are too small[5] and the above construction needs to be modified as follows. For each $r \geq 0$, $t \in \mathbb{Z}$ and $X = E, F, \Theta_1$ or Θ_2, consider the element

$$X_{r;t} = \sum_{s=0}^{r}(-1)^s\begin{pmatrix} r \\ s\end{pmatrix}X_{s+t}$$

where $\Theta_{i,l} = (\Theta_{i,l}^+ - \Theta_{i,l}^-)/(q-q^{-1})$. Note that $X_{r;t} = x\otimes z^t(1-z)^r \mod \hbar$ where $x \in \mathfrak{g}$ is such that $X = x \mod \hbar$. Let $\mathcal{K}_r$ be the two–sided ideal of $U_\hbar(L\mathfrak{gl}_2)$

[5]for example, $I\otimes(z-1)^2 \notin \bigcup_{n\geq 1} J_1^n$, where $I = E_{11} + E_{22}$ is the identity matrix.

generated by the elements $\{X_{r';t}\}_{r'\geq r,t\in\mathbb{Z}}$, and $\hbar$ if $r=1$. Finally, let $\mathcal{J}_n\subset U_\hbar(L\mathfrak{gl}_2)$ be the ideal

$$\mathcal{J}_n=\sum_{\substack{n_1,\ldots,n_k\geq 1\\ n_1+\cdots+n_k=n}}\mathcal{K}_{n_1}\cdots\mathcal{K}_{n_k}$$

Then, $\mathcal{J}_n$ is a descending filtration, $\mathcal{J}_n\mathcal{J}_m\subset\mathcal{J}_{n+m}$, and the completion

$$\widehat{U_\hbar(L\mathfrak{gl}_2)}=\lim_{\longleftarrow}U_\hbar(L\mathfrak{gl}_2)/\mathcal{J}_n$$

is a flat deformation of $\widehat{U(L\mathfrak{gl}_2)}$.

Remark. Note that $\hbar\in\mathcal{K}_1$ implies that $\hbar\mathcal{J}_n\subset\mathcal{J}_{n+1}$ for any $n\geq 0$.

9.8.

Proposition.

(1) *The center of* $\widehat{U_\hbar(L\mathfrak{sl}_2)}$ *is trivial.*
(2) *The center of* $\widehat{U_\hbar(L\mathfrak{gl}_2)}$ *is generated by the elements* $\mathfrak{z}_r=q^rD_{1,r}+q^{-r}D_{2,r}$, $r\in\mathbb{N}$.

Proof. (1) follows from the fact that $\widehat{U_\hbar(L\mathfrak{sl}_2)}$ is a flat deformation of $\widehat{U(L\mathfrak{sl}_2)}$ and that the latter algebra has trivial center.

(2) By definition of the series $\mathrm{qdet}^\pm(z)$,

$$\mathrm{qdet}^\pm(z)=\Theta_1^+(q^{-1}z)\Theta_2^+(qz)=q^{\pm(D_{1,0}+D_{2,0})}\exp\left(\pm(q-q^{-1})\sum_{r\geq 1}\mathfrak{z}_rz^{-r}\right)\quad(9.5)$$

Thus, the center $Z(U_\hbar(L\mathfrak{gl}_2))$ is generated by the elements $\mathfrak{z}_r$, $r\in\mathbb{Z}$. The fact that its completion is generated by the $\mathfrak{z}_r$, $r\in\mathbb{N}$ follows from the analogous statement in the classical case. □

9.9. The following is straighforward and will henceforth be used implicitly.

Lemma. *If* $\zeta\in 1+\hbar\mathbb{C}[[\hbar]]$, *the evaluation homomorphism* ev_ζ *extends to* $\widehat{U_\hbar(L\mathfrak{gl}_2)}\to U_\hbar\mathfrak{gl}_2$.

9.10. The operators $\mathbb{L}_1,\mathbb{L}_2$. Define, for any $r\geq 0$ and $i=1,2$

$$\begin{aligned}\widetilde{D}_{i,r}&=D_{i,0}+\sum_{s=1}^r(-1)^s\binom{r}{s}\frac{s}{[s]}D_{i,s}\\ \widetilde{H}_r&=H_0\;\;+\sum_{s=1}^r(-1)^s\binom{r}{s}\frac{s}{[s]}H_s\end{aligned}\quad(9.6)$$

The proof of the following result is given in Appendix B.

Proposition. *The elements* $\widetilde{D}_{1,r},\widetilde{D}_{2,r}$ *lie in* $\mathcal{J}_r$. *Similarly,* $\widetilde{H}_r\in\mathcal{J}^r$ *for any* $r\in\mathbb{N}$.

Thus, the following are well defined elements of $\widehat{U_\hbar(L\mathfrak{gl}_2)}$ and $\widehat{U_\hbar(L\mathfrak{sl}_2)}$ respectively.

$$\mathbb{L}_1 = q^{-D_1} \exp\left(\sum_{r\geq 1} \frac{\widetilde{D}_{1,r}}{r}\right) \qquad \mathbb{L}_2 = q^{-D_1} \exp\left(\sum_{r\geq 1} \frac{\widetilde{D}_{2,r}}{r}\right) \tag{9.7}$$

$$\mathbb{L} = \exp\left(\sum_{r\geq 1} \frac{\widetilde{H}_r}{r}\right) = \mathbb{L}_1\mathbb{L}_2^{-1} \tag{9.8}$$

9.11.

PROPOSITION. *The following holds on $\widehat{U_\hbar(L\mathfrak{gl}_2)}$ for $i = 1, 2$*

$$L_i = \mathrm{Ad}(\mathbb{L}_i) \qquad \text{and} \qquad L = \mathrm{Ad}(\mathbb{L})$$

PROOF. It is clear that $\mathrm{Ad}(\mathbb{L}_i)$ fix $\Theta_j^\pm(z)$. By Lemma 8.3

$$[\widetilde{D}_{1,r}, E(z)] = (1 - q^{-1}z)^r E(z) \qquad \text{and} \qquad [\widetilde{D}_{2,r}, E(z)] = -(1 - qz)^r E(z)$$

which shows that conjugation with $\overline{\mathbb{L}}_i = \exp\left(\sum_{r\geq 1} \frac{\widetilde{D}_{i,r}}{r}\right)$ is given by

$$\mathrm{Ad}\left(\overline{\mathbb{L}}_1\right) E(z) = qz^{-1}E(z) \qquad \text{and} \qquad \mathrm{Ad}\left(\overline{\mathbb{L}}_2\right) E(z) = qzE(z)$$

The relations $\mathrm{Ad}(\mathbb{L}_1)E(z) = z^{-1}E(z)$ and $\mathrm{Ad}(\mathbb{L}_2)E(z) = zE(z)$ follow from this computation and the fact that $[D_1, E(z)] = E(z)$. The remaining relations are proved analogously. □

COROLLARY. *The product $\mathbb{S}_0\mathbb{S}_1$ is equal to $\mathbb{L}$.*

PROOF. Propositions 9.5 and 9.11 imply that

$$\mathrm{Ad}(\mathbb{S}_0\mathbb{S}_1) = T_0T_1 = L_1L_2^{-1} = \mathrm{Ad}(\mathbb{L})$$

The result now follows from the fact that $\widehat{U_\hbar(L\mathfrak{sl}_2)}$ has trivial center by Proposition 9.8. □

9.12. Let $\mathcal{X}_+, \mathcal{X}_- \subset U_\hbar(L\mathfrak{gl}_2)$ be the left ideals generated by $\{E_k\}_{k\in\mathbb{Z}}$ and $\{F_k\}_{k\in\mathbb{Z}}$ respectively.

PROPOSITION. *The elements $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}$ are grouplike modulo the subspace*

$$\mathcal{N} = \mathcal{X}_+ \otimes \mathcal{X}_- + \mathcal{X}_- \otimes \mathcal{X}_+$$

PROOF. Let $\mathfrak{z}_r \in Z(U_\hbar(L\mathfrak{gl}_2))$ be the elements defined by (9.5). Since $\mathrm{qdet}^+(z)$ is grouplike, the elements $\mathfrak{z}_r$ are primitive. By [**6**, Prop. 4.4 (iii)], the elements $H_r = D_{1,r} - D_{2,r}$ are primitive modulo $\mathcal{N}$. The same therefore holds for $D_{j,r}$ and hence for $\widetilde{D}_{j,r}$, which implies the desired assertion. □

REMARK. An alternative proof of Proposition 9.12 for the element $\mathbb{L}$ can be obtained using the results of [**19, 20**]. Recall that for a symmetrisable Kac–Moody algebra $\mathfrak{g}(\mathbf{A})$ and a node i of its Dynkin diagram, the corresponding quantum Weyl group element $\mathbb{S}_i$ satisfies

$$\Delta(\mathbb{S}_i) = \mathcal{R}_{i,0}^{21} (\mathbb{S}_i \otimes \mathbb{S}_i)$$

where $\mathcal{R}_{i,0}$ is the truncated R–matrix of $U_\hbar\mathfrak{sl}_2^{(i)} \subset U_\hbar\mathfrak{g}(\mathbf{A})$. Thus the elements $\mathbb{S}_i$ are group–like modulo $\mathcal{N}$ and hence the same is true for $\mathbb{L} = \mathbb{S}_0\mathbb{S}_1$.

9.13. Action on highest weight vectors. For any $\lambda \in \mathbb{N}$, let $\mathcal{V}_\lambda$ be the $(\lambda+1)$–dimensional, indecomposable representation of $U_\hbar\mathfrak{gl}_2$ with highest weight vector Ω_λ such that $D_1\Omega_\lambda = \lambda\Omega_\lambda$ and $D_2\Omega_\lambda = 0$. Let $\overline{\Omega}_\lambda$ be its lowest weight vector, and $\mathcal{V}_\lambda(\zeta)$ the corresponding evaluation representation of $U_\hbar(L\mathfrak{gl}_2)$, where $\zeta \in 1+\hbar\mathbb{C}[[\hbar]]$. We compute below the action of the operators $\mathbb{L}_1, \mathbb{L}_2$ on the highest and lowest weight vectors

$$\Omega = \bigotimes_{i=1}^k \Omega_{\lambda_i} \in \mathcal{V}_{\lambda_1}(\zeta_1)\otimes\cdots\otimes\mathcal{V}_{\lambda_k}(\zeta_k) \ni \bigotimes_{i=1}^k \overline{\Omega}_{\lambda_i} = \overline{\Omega}$$

of a tensor product of these evaluation representations. We shall need the following

LEMMA. *The following holds on* $\mathcal{V}_\lambda(\zeta)$

$$\Theta_1^\pm(z)\,\Omega_\lambda = q^\lambda \frac{z-q^{-\lambda-1}\zeta}{z-q^{\lambda-1}\zeta}\,\Omega_\lambda \qquad \Theta_2^\pm(z)\,\Omega_\lambda = \Omega_\lambda$$

$$\Theta_1^\pm(z)\,\overline{\Omega}_\lambda = \overline{\Omega}_\lambda \qquad \Theta_2^\pm(z)\,\overline{\Omega}_\lambda = q^\lambda \frac{z-q^{-\lambda+1}\zeta}{z-q^{\lambda+1}\zeta}\,\overline{\Omega}_\lambda$$

PROOF. By [**6**, §4.2],

$$\psi^\pm(z)\,\Omega_\lambda = q^\lambda \frac{z-q^{-\lambda-1}\zeta}{z-q^{\lambda-1}\zeta}\,\Omega_\lambda \qquad \text{and} \qquad \psi^\pm(z)\,\overline{\Omega}_\lambda = q^{-\lambda} \frac{z-q^{\lambda+1}\zeta}{z-q^{-\lambda+1}\zeta}\,\overline{\Omega}_\lambda$$

By §8.6, the series $\mathrm{qdet}^\pm(z)$ acts on $\mathcal{V}_\lambda(\zeta)$ as multiplication by $q^\lambda \frac{z-q^{-\lambda}\zeta}{z-q^\lambda\zeta}$. Using $\Psi^\pm(z) = \Theta_1^\pm(z)/\Theta_2^\pm(z)$ and $\mathrm{qdet}^\pm(z) = \Theta_1^\pm(q^{-1}z)\Theta_2^\pm(qz)$ shows that

$$\Theta_2^\pm(qz)\Theta_2^\pm(q^{-1}z)\Omega_\lambda = \Omega_\lambda \qquad \text{and} \qquad \Theta_1^\pm(qz)\Theta_1^\pm(q^{-1}z)\overline{\Omega}_\lambda = \overline{\Omega}_\lambda$$

from which the stated formulae readily follow. □

PROPOSITION. *The following holds on* $\mathcal{V}_{\lambda_1}(\zeta_1)\otimes\cdots\otimes\mathcal{V}_{\lambda_k}(\zeta_k)$

$$\mathbb{L}_1\,\Omega = \prod_{1\le a\le k} \zeta_a^{-\lambda_a}\,\Omega \qquad \mathbb{L}_2\,\Omega = \prod_{1\le a\le k} q^{-\lambda_a}\,\Omega$$

$$\mathbb{L}_1\,\overline{\Omega} = \overline{\Omega} \qquad \mathbb{L}_2\,\overline{\Omega} = \prod_{1\le a\le k} q^{-\lambda_a}\zeta_a^{-\lambda_a}\,\overline{\Omega}$$

PROOF. By Proposition 9.12, it suffices to prove the result for $k=1$. We consequently drop the subscript a from the computations below. By Lemma 9.13, $D_{2,r}\Omega = 0$ for any $r\ge 0$, $D_{1,0}\Omega = \lambda\Omega$ and, for $r\ge 1$,

$$D_{1,r}\Omega = q^{-r}\frac{[\lambda r]}{r}\zeta^r\Omega$$

This implies that $\widetilde{D}_{2,r}\Omega = 0$ and

$$\widetilde{D}_{1,r}\Omega = \left(\sum_{t=0}^{\lambda-1}\left(1-\zeta q^{2t-\lambda}\right)^r\right)\Omega$$

Thus, $\mathbb{L}_2\Omega = q^{-D_1}\Omega = q^{-\lambda}\,\Omega$ and $\mathbb{L}_1\Omega = q^{-D_1}(\zeta^{-\lambda}q^\lambda)\Omega = \zeta^{-\lambda}\Omega$ as claimed. The remaining relations follows similarly. □

9.14. Braid relations. We now show that the operators $\mathbb{S}_1, \mathbb{L}_1, \mathbb{L}_2$ satisfy relations very similar to those defining the affine braid group B_{GL_2}.

LEMMA. *Let $\mathcal{V} = \mathcal{V}_1$ be the standard two–dimensional representation of $U_\hbar\mathfrak{gl}_2$. Then, the evaluation representations*

$$\mathcal{V}(\underline{\zeta}) = \mathcal{V}(\zeta_1) \otimes \cdots \otimes \mathcal{V}(\zeta_k) \tag{9.9}$$

separate the elements of the centre of $\widehat{U_\hbar(L\mathfrak{gl}_2)}$.

PROOF. It follows from §8.6, and the fact that the series $\mathrm{qdet}^{\pm}(z)$ are grouplike that their action on $\mathcal{V}(\zeta)$ is given by multiplication by

$$q^{\pm k} \prod_{a=1}^{k} \left(\frac{1 - q^{\mp 1}\zeta_a^{\pm 1} z^{\mp 1}}{1 - q^{\pm 1}\zeta_a^{\pm 1} z^{\mp 1}} \right)$$

Thus, the generators $\mathfrak{z}_r$, $r \in \mathbb{N}$ of $Z(U_\hbar(L\mathfrak{gl}_2))$ defined by (9.5) act as multiplication by the power sums

$$\mathfrak{z}_r = \frac{[r]}{r} \sum_{a=1}^{k} \zeta_a^r$$

The claim now follows from the fact that these are algebraically independent. □

THEOREM. *The elements $\mathbb{S}_1, \mathbb{L}_1, \mathbb{L}_2$ satisfy the following relations*

(1) $\mathbb{L}_1\mathbb{L}_2 = \mathbb{L}_2\mathbb{L}_1$.
(2) $\mathbb{S}_1\mathbb{L}_2\mathbb{S}_1 = (-1)^I\mathbb{L}_1$.

where $I = D_{1,0} + D_{2,0}$.

PROOF. The first assertion is obvious since $\mathbb{L}_1, \mathbb{L}_2$ are defined in terms of the commutating elements $D_{i,r}$. By Propositions 9.5 and 9.11, both sides of (2) define the same automorphism of $U_\hbar(L\mathfrak{gl}_2)$ and therefore agree up a central element c, $\mathbb{S}_1\mathbb{L}_2\mathbb{S}_1 = c\mathbb{L}_1$. To determine c it suffices, by Lemma 9.14, to compute it on all evaluation representations (9.9). Let $\Omega, \overline{\Omega}$ be the highest and lowest weight vectors in $\mathcal{V}(\Omega)$. By [**20**]

$$\mathbb{S}_1\Omega = (-1)^k q^k \overline{\Omega} \qquad \text{and} \qquad \mathbb{S}_1\overline{\Omega} = \Omega$$

Together with Proposition 9.13, this implies that

$$\mathbb{S}_1\mathbb{L}_2\mathbb{S}_1\overline{\Omega} = \mathbb{S}\mathbb{L}_2\Omega = q^{-k}\mathbb{S}\Omega = (-1)^k\overline{\Omega} = (-1)^k\mathbb{L}_1\overline{\Omega}$$

so that c acts as $(-1)^I$ on $\mathcal{V}(\underline{\zeta})$ as claimed. □

9.15. The quantum Weyl group of $U_\hbar(L\mathfrak{gl}_2)$. Set

$$\mathbb{S} = \mathbb{S}_1(-1)^{D_1} = (-1)^{D_2}\mathbb{S}_1 \tag{9.10}$$

By Theorem 9.14, the elements $\mathbb{S}, \mathbb{L}_1, \mathbb{L}_2$ satisfy the defining relations of B_{GL_2}, namely

$$\mathbb{L}_1\mathbb{L}_2 = \mathbb{L}_2\mathbb{L}_1 \qquad \text{and} \qquad \mathbb{S}\mathbb{L}_2\mathbb{S} = \mathbb{L}_1$$

We shall refer to $\mathbb{S}, \mathbb{L}_1, \mathbb{L}_2$ as the *quantum Weyl group elements* of $U_\hbar(L\mathfrak{gl}_2)$. Note that the fact that the element $\mathbb{S}$ differs from $\mathbb{S}_1$ by the sign $(-1)^{D_1}$ is in agreement with the fact that their classical limits are, respectively

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which are the generators of the (Tits extensions of the) Weyl groups of GL_2 and SL_2.

10. The dual pair $(U_\hbar\mathfrak{gl}_k, U_\hbar\mathfrak{gl}_n n)$

In this section we review a deformation of the matrix space $\mathbb{C}[\mathcal{M}_{k,n}]$ as a joint representation space for $U_\hbar\mathfrak{gl}_k$ and $U_\hbar\mathfrak{gl}_n$. The main reference for this section is [**26**, §5].

10.1. Quantum matrix $(k\times n)$ **space.** By definition, $\mathbb{C}_\hbar[\mathcal{M}_{k,n}]$ is the algebra over $\mathbb{C}[[\hbar]]$ topologically generated by elements $\{X_{ai}$-$1 \leq a \leq k, 1 \leq i \leq n$ subject to the relations

$$X_{ai}X_{bj} = \begin{cases} X_{bj}X_{ai} & \text{if } a<b \text{ and } i>j \text{ or } a>b \text{ and } i<j \\ q^{-1}X_{bj}X_{ai} & \text{if } a=b \text{ and } i<j \text{ or } a<b \text{ and } i=j \\ X_{bj}X_{ai}-(q-q^{-1})X_{bi}Xaj & \text{if } a>b \text{ and } i>j \end{cases}$$

For each $m=(m_{ai})_{1\leq a\leq k;1\leq i\leq n}$ define

$$\begin{aligned} X^m &= (X_{11}^{m_{11}}\cdots X_{k1}^{m_{k1}})\cdots(X_{1n}^{m_{1n}}\cdots X_{kn}^{m_{kn}}) \\ &= (X_{11}^{m_{11}}\cdots X_{1n}^{m_{1n}})\cdots(X_{k1}^{m_{k1}}\cdots X_{kn}^{m_{kn}}) \end{aligned}$$

Then, the set $\{X^m\}_{m\in\mathcal{M}_{k\times n}(\mathbb{N})}$ is a basis for $\mathbb{C}_\hbar[\mathcal{M}_{k,n}]$ over $\mathbb{C}[[\hbar]]$.

10.2. The joint action of $(U_\hbar\mathfrak{gl}_k, U_\hbar\mathfrak{gl}_n)$**.** Define the following operators on $\mathbb{C}_\hbar[\mathcal{M}_{k,n}]$, for each $b\in\{1,\dots,k\}$ and $a\in\{1,\dots,k-1\}$:

$$\begin{aligned} D_b^{(k)}X^m &= \sum_i m_{bi}X^m \\ E_a^{(k)}X^m &= \sum_{i=1}^n [m_{a+1,i}] \prod_{j=i+1}^n q^{(m_{aj}-m_{a+1,j})}X^{m+\varepsilon_{ai}-\varepsilon a+1,i} \\ F_a^{(k)}X^m &= \sum_{i=1}^n [m_{ai}] \prod_{j=1}^{i-1} q^{-(m_{aj}-m_{a+1,j})}X^{m-\varepsilon_{aj}+\varepsilon_{a+1,j}} \end{aligned}$$

Similarly define the operators for each $j\in\{1,\dots,n\}$ and $i\in\{1,\dots,n-1\}$:

$$\begin{aligned} D_j^{(n)}X^m &= \sum_{a=1}^k m_{aj}X^m \\ E_i^{(n)}X^m &= \sum_{a=1}^k [m_{a,i+1}] \prod_{b=a+1}^k q^{m_{b,i}-m_{b,i+1}}X^{m+\varepsilon_{ai}-\varepsilon_{a,i+1}} \\ F_i^{(n)}X^m &= \sum_{a=1}^k [m_{ai}] \prod_{b=1}^{a-1} q^{-(m_{b,i}-m_{b,i+1})}X^{m-\varepsilon_{ai}+\varepsilon_{a,i+1}} \end{aligned}$$

The following result is proved in [**26**, Thm. 5.4] and builds upon the approach to quantum matrix space described in [**1**].

THEOREM. *The operators above define a structure of an algebra module on* $\mathbb{C}_\hbar[\mathcal{M}_{k,n}]$ *over* $U_\hbar\mathfrak{gl}_k\otimes U_\hbar\mathfrak{gl}_n$*. Moreover as a* $U_\hbar\mathfrak{gl}_k$ *(resp.* $U_\hbar\mathfrak{gl}_n$*) module we have*

$$\mathbb{C}_\hbar[\mathcal{M}_{k,n}] \cong \mathbb{C}_\hbar[\mathcal{M}_{k,1}]^{\otimes n} \ \left(\textit{resp. } \mathbb{C}_\hbar[\mathcal{M}_{1,n}]^{\otimes k}\right)$$

11. Affine braid group actions on quantum matrix space

11.1. In this section, we compare two actions of the affine braid group B_{GL_2} on the quantum matrix space $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$ described in Section 10. The first is described in 7.4 and arises by regarding $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$ as the $U_\hbar\mathfrak{gl}_k$–module $\mathbb{C}_\hbar[\mathcal{M}_{k,1}]^{\otimes 2}$. It is given in the generators $b, \mathcal{L}_1, \mathcal{L}_2$ of Section 4.4 by

$$b \mapsto (1\,2)\mathcal{R} \qquad \mathcal{L}_1 \mapsto (q^{2s})^{(1)}\mathcal{R} \qquad \mathcal{L}_2 \mapsto \mathcal{R}^{-1}(q^{2s})^{(2)}$$

and depends upon the choice of a diagonal matrix

$$s = \sum_{a=1}^{k} s_a E_{aa} \in \mathfrak{gl}_k$$

The second action is obtained by regarding $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$ as the tensor product of evaluation representations of $U_\hbar(L\mathfrak{gl}_2)$

$$\mathbb{C}_\hbar[\mathcal{M}_{k,2}] \cong \mathbb{C}_\hbar[\mathcal{M}_{1,2}](\zeta_1) \otimes \cdots \otimes \mathbb{C}_\hbar[\mathcal{M}_{1,2}](\zeta_k)$$

corresponding to a choice of evaluation points $\underline{\zeta} = (\zeta_1, \ldots, \zeta_k) \in (1 + \hbar\mathbb{C}[[\hbar]])^k$. It is given in terms of the quantum Weyl group elements of $U_\hbar(L\mathfrak{gl}_2)$ defined in 9.15 by

$$b \mapsto \mathbb{S} \qquad \mathcal{L}_1 \mapsto \mathbb{L}_1 \qquad \mathcal{L}_2 \mapsto \mathbb{L}_2$$

It was shown in [**26**] that the restrictions of these actions to the braid group $B \subset B_{GL_2}$ generated by b essentially coincide. Specifically, one has $(12)\mathcal{R}_{\mathfrak{sl}_k} = \mathbb{S}_1 q^{-(D_1 + D_1 D_2/k)}(-1)^{D_1}$ [**26**, Thm. 6.5] which, by Remark 7.1 and (9.10), implies that

$$(1\,2)\mathcal{R} = \mathbb{S}\, q^{-D_1} \tag{11.1}$$

The result below shows that similar relations hold between the operators giving the actions of the generators $\mathcal{L}_1, \mathcal{L}_2$.

THEOREM. *Assume that the evaluation points $\zeta_1, \ldots, \zeta_k$ are given by*

$$\zeta_a = q^{-2s_a} \tag{11.2}$$

Then, the following holds on $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$

$$(q^{2s})^{(1)}\mathcal{R} = \mathbb{L}_1 \tag{11.3}$$

$$\mathcal{R}^{-1}(q^{2s})^{(2)} = \mathbb{L}_2\, q^I \tag{11.4}$$

PROOF. It is easy to see, using $\mathcal{L}_2 = b^{-1}\mathcal{L}_1 b^{-1}$ and $\mathbb{L}_2 = \mathbb{S}^{-1}\mathbb{L}_1\mathbb{S}^{-1}$ that (11.1) and (11.3) imply (11.4). The proof of (11.3) occupies the rest of this section. We first show in Proposition 11.4 that both sides of (11.3) have the same commutation relation with elements in $U_\hbar(L\mathfrak{sl}_2)$. We then check in Lemma 11.5 that they coincide on the tensor product of highest weight vectors in

$$\mathbb{C}_\hbar[\mathcal{M}_{1,2}][\lambda_1] \otimes \cdots \otimes \mathbb{C}_\hbar[\mathcal{M}_{1,2}][\lambda_k] \subset \mathbb{C}_\hbar[\mathcal{M}_{k,2}]$$

where the notation $[\lambda_i]$ refers to the homegeneity degree in the variables X_{i1}, X_{i2}. If the evaluation points are generic, the statement follows because the action of $U_\hbar(L\mathfrak{sl}_2)$ on the above tensor product is irreducible. The general case follows by continuity. □

11.2. Let $\tau = (1\,2)$ be the flip acting on $\mathbb{C}_\hbar[\mathcal{M}_{k,2}] \cong \mathbb{C}_\hbar[\mathcal{M}_{k,1}]^{\otimes 2}$. In terms of the monomial basis $\{X^m\}$, the action of τ is given by

$$(X_{11}^{m_{11}} X_{12}^{m_{12}}) \cdots (X_{k1}^{m_{k1}} X_{k2}^{m_{k2}}) \mapsto (X_{11}^{m_{12}} X_{12}^{m_{11}}) \cdots (X_{k1}^{m_{k2}} X_{k2}^{m_{k1}})$$

LEMMA. *The following holds on* $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$

(1) $(q^{2s})^{(1)} = q^{2s_1 D_1} \otimes \cdots \otimes q^{2s_k D_1}$.

(2) *For any* $x \in U_\hbar \mathfrak{gl}_2^{\otimes k}$

$$\mathrm{Ad}\,(\tau)\; x = \theta^{\otimes k}(x)$$

where $\theta \in \mathrm{Aut}(U_\hbar \mathfrak{gl}_2)$ *is the involution given by*

$$D_1 \leftrightarrow D_2 \qquad \text{and} \qquad E \leftrightarrow F$$

PROOF. (1) and (2) follow from the formulae giving the action of $U_\hbar\mathfrak{gl}_2$ in 10.2. □

11.3.

LEMMA. *Assume that the evaluation points for* $U_\hbar(L\mathfrak{sl}_2)$ *are given by* (11.2). *Then, the following holds on* $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$ *for any* $X \in U_\hbar(L\mathfrak{sl}_2)$

$$\mathrm{Ad}\left((q^{2s})^{(1)}\,\tau\right)\, X = \mathrm{Ad}(q^{\mathcal{H}/2})\omega(X)$$

where $\omega \in \mathrm{Aut}(U_\hbar(L\mathfrak{sl}_2))$ *is the diagram automorphism defined in* 8.8.

PROOF. The stated identity clearly holds for $X = \mathcal{H}$. It therefore suffices to check it on the remaining generators E_0, F_0, E_{-1}, F_{-1} of $U_\hbar(L\mathfrak{sl}_2)$. Moreover, since

$$\left((q^{2s})^{(1)}\,\tau\right)^2 = q^{2s} \otimes q^{2s}$$

commutes with the action of $U_\hbar(L\mathfrak{sl}_2)$, $\mathrm{Ad}((q^{2s})^{(1)}\,\tau)$ acts as an involution on the image of $U_\hbar(L\mathfrak{sl}_2)$. Since so does $\mathrm{Ad}(q^{\mathcal{H}/2})\omega$, it suffices to check the identity on only one half of these generators which, in view of the formulae (8.5) can be taken to be E_0, F_0.

By (8.3), E_0 acts on $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$ by

$$\mathrm{ev}_{\underline{\zeta}}\,\Delta^{(k)}(E_0) = \sum_{a=1}^{k} 1^{\otimes(a-1)} \otimes E \otimes K^{\otimes(k-a)}$$

so that, by Lemma 11.2

$$\mathrm{Ad}\left((q^{2s})^{(1)}\,\tau\right) \mathrm{ev}_{\underline{\zeta}}\,\Delta^{(k)}(E_0) = \sum_{a=1}^{k} 1^{\otimes(a-1)} \otimes q^{-2s_a} F \otimes (K^{-1})^{\otimes(k-a)}$$

On the other hand, the k–fold coproduct of $\mathrm{Ad}(q^{\mathcal{H}/2})\omega(E_0) = q^{-1}K^{-1}F_1$ is equal to

$$(K^{-1})^{\otimes k} \sum_{a=1}^{k} K^{\otimes(a-1)} \otimes q^{-1}F_1 \otimes 1^{\otimes(k-a)} = \sum_{a=1}^{k} 1^{\otimes(a-1)} \otimes q^{-1}K^{-1}F_1 \otimes (K^{-1})^{\otimes(k-a)}$$

so that its image under $\mathrm{ev}_{\underline{\zeta}}$ is equal to

$$\sum_{a=1}^{k} 1^{\otimes(a-1)} \otimes q^{-2}\zeta_a K^{-1}FK \otimes (K^{-1})^{\otimes(k-a)} = \sum_{a=1}^{k} 1^{\otimes(a-1)} \otimes \zeta_a F \otimes (K^{-1})^{\otimes(k-a)}$$

The computation for F_0 is identical. □

11.4.

Proposition. *Assume that the evaluation points are given by* (11.2). *Then, the following holds on* $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$ *for any* $X \in U_\hbar(L\mathfrak{sl}_2)$

$$\mathrm{Ad}\left((q^{2s})^{(1)}\mathcal{R}\right) X = \mathrm{Ad}(\mathbb{L}_1)X$$

Proof. By (11.1) and Lemma 11.3, the left–hand side is equal to

$$\begin{aligned}\mathrm{Ad}\left((q^{2s})^{(1)}\tau\,\mathbb{S}_1 q^{-D_1}(-1)^{D_1}\right) X &= \mathrm{Ad}(q^{-D_1}(-1)^{D_1}q^{\mathcal{H}/2})\omega\,T_1(X)\\ &= \mathrm{Ad}(q^{-I/2}(-1)^{D_1})T_0\,\omega(X)\\ &= L_1(X)\\ &= \mathrm{Ad}(\mathbb{L}_1)X\end{aligned}$$

where the second equality uses (9.1), the third one Remark 9.5, and the last one Proposition 9.11. □

11.5. Let $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$ and set

$$\Omega = X_{11}^{\lambda_1} X_{21}^{\lambda_2} \cdots X_{k1}^{\lambda_k} \in \mathbb{C}_\hbar[\mathcal{M}_{k,2}]$$

Lemma. *The following holds*

$$(q^{2s})^{(1)}\mathcal{R}\,\Omega = \prod_{a=1}^{k} q^{2s_a\lambda_a}\,\Omega \qquad \textit{and} \qquad \mathbb{L}_1\,\Omega = \prod_{a=1}^{k}\zeta_a^{-\lambda_a}\,\Omega$$

Proof. Under the identification $\mathbb{C}_\hbar[\mathcal{M}_{k,2}] \cong \mathbb{C}_\hbar[\mathcal{M}_{k,1}]^{\otimes 2}$ of $U_\hbar\mathfrak{gl}_k$–modules, Ω is the tensor product of a vector in the q–deformation of $S^{\sum_a \lambda_a}\mathbb{C}^k$ and a vector in the the trivial representation of $U_\hbar\mathfrak{gl}_k$. Thus $\mathcal{R}\Omega = \Omega$, which implies the first stated formula. Under the identification $\mathbb{C}_\hbar[\mathcal{M}_{k,2}] \cong \mathbb{C}_\hbar[\mathcal{M}_{1,2}]^{\otimes k}$ of $U_\hbar\mathfrak{gl}_2$–modules, Ω is the tensor product of highest weight vectors in $\mathcal{V}_{\lambda_1} \otimes \cdots \otimes \mathcal{V}_{\lambda_k}$, where the notation is as in 9.13. The result then follows from Proposition 9.13. □

12. Monodromy theorems

For any $\lambda \in \mathbb{N}$, denote by $V_\lambda = S^\lambda\mathbb{C}^2$ the λth symmetric power of the defining representation of $\mathfrak{gl}_2$, and by $\mathcal{V}_\lambda$ its quantum deformation, that is the finite–dimensional $U_\hbar\mathfrak{gl}_2$–module such that $\mathcal{V}_\lambda/\hbar\mathcal{V}_\lambda \cong V_\lambda$ and I acts as multiplication by λ on $\mathcal{V}_\lambda$.

Fix now $\underline{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$, $\underline{s} = (s_1, \dots, s_k) \in \mathbb{C}^k$, and denote by

$$V_{\underline{\lambda}}(\underline{s}) = V_{\lambda_1}(s_1) \otimes \cdots \otimes V_{\lambda_k}(s_k)$$

the tensor product of the evaluation modules of the Yangian $Y_{\mathsf{h}}\mathfrak{gl}_2$ corresponding to the points $r_a = \mathsf{h}(s_a + \frac{I+1}{2})$. By Lemma 3.4, the restriction of $V_{\underline{\lambda}}(\underline{s})$ to $Y_{\mathsf{h}}\mathfrak{sl}_2 \subset Y_{\mathsf{h}}\mathfrak{gl}_2$ is the tensor product of the modules $V_{\lambda_1}, \dots, V_{\lambda_k}$ evaluated at the points $\mathsf{h}s_1, \dots, \mathsf{h}s_k$. Denote by

$$\mathcal{V}_{\underline{\lambda}}(\underline{\zeta}) = \mathcal{V}_{\lambda_1}(\zeta_1) \otimes \cdots \otimes \mathcal{V}_{\lambda_k}(\zeta_k)$$

the tensor product of evaluation modules of the quantum loop algebra $U_\hbar(L\mathfrak{gl}_2)$ corresponding to the evaluation points $(\zeta_1, \dots, \zeta_k) \in (\mathbb{C}[[\hbar]]^\times)^k$. The following is the main result of this paper.

THEOREM. *Let $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{gl}_2$. Assume that $\hbar = 4\pi\imath\mathsf{h}$, and that $\zeta_a = \exp(-\hbar s_a)$ for any a. Then, the monodromy of the trigonometric Casimir connection of $\mathfrak{g}$ on $V_{\underline{\lambda}}(\underline{s})$ is described by the quantum Weyl operators of the quantum loop algebra $U_\hbar(L\mathfrak{g})$ on $\mathcal{V}_{\underline{\lambda}}(\underline{s})$.*

PROOF. We first prove the result for $\mathfrak{g} = \mathfrak{gl}_2$. Let $\mathbb{C}[\mathcal{M}_{k,2}]$ be the space of functions on the space of $k \times 2$ matrices described in Section 6. As a $U\mathfrak{gl}_2^{\otimes k}$–module, $V_{\underline{\lambda}}(\underline{s})$ may be realised as the subspace of $\mathbb{C}[\mathcal{M}_{k,2}]$ via

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k} \subset S^\bullet\mathbb{C}^2 \otimes \cdots \otimes S^\bullet\mathbb{C}^2 \cong \mathbb{C}[\mathcal{M}_{k,2}]$$

Combining the duality statement of Corollary 6.5 with the computation of the monodromy of the trigonometric KZ connection for $\mathfrak{gl}_k$ on $n = 2$ points given in 7.4 and Theorem 11.1, we see that the monodromy of the trigonometric connection of $\mathfrak{gl}_2$ on $V_{\underline{\lambda}}(\underline{s})$ and the quantum Weyl group operators $\mathbb{S}, \mathbb{L}_1, \mathbb{L}_2$ giving the action of B_{GL_2} on the quantum matrix space $\mathbb{C}_\hbar[\mathcal{M}_{k,2}]$ are related by

$$\pi_{C,\underline{s}}(b) = \pi_{\mathrm{KZ},s}(b)q^{I/2} = (1\,2)\mathcal{R}\,q^{I/2} = \mathbb{S}\,q^{-\mathcal{H}/2}$$
$$\pi_{C,\underline{s}}(\mathcal{L}_1) = \pi_{\mathrm{KZ},s}(\mathcal{L}_1) = (q^{2s})^{(1)}\mathcal{R} = \mathbb{L}_1$$
$$\pi_{C,\underline{s}}(\mathcal{L}_2) = \pi_{\mathrm{KZ},s}(\mathcal{L}_2)q^{-I} = \mathcal{R}^{-1}(q^{2s})^{(2)}\,q^{-I} = \mathbb{L}_2$$

where $b, \mathcal{L}_1, \mathcal{L}_2$ are the generators of B_{GL_2} described in 4.5. The assertion of the theorem now follows since $\mathbb{S}\,q^{-\mathcal{H}/2} = q^{\mathcal{H}/4}\,\mathbb{S}\,q^{-\mathcal{H}/4}$. For $\mathfrak{g} = \mathfrak{sl}_2$, the corresponding braid group is generated by $b, \mathcal{L} = \mathcal{L}_1\mathcal{L}_2^{-1}$. Moreover,

$$\pi_{C,\underline{s}}(\mathcal{L}) = \pi_{C,\underline{s}}(\mathcal{L}_1)\pi_{C,\underline{s}}(\mathcal{L}_2)^{-1} = \mathbb{L}_1\mathbb{L}_2^{-1} = \mathbb{L}$$

and

$$\pi_{C,\underline{s}}(b) = \pi_{C,\underline{s}}(b)(-1)^{E_{11}} = \mathbb{S}(-1)^{D_1}\,q^{-\mathcal{H}/2} = \mathbb{S}\,q^{-\mathcal{H}/2}$$

□

Appendix A. Monodromy of the trigonometric KZ equations (after Etingof–Geer–Schiffmann)

This appendix follows [**12**] closely. It only differs from it in the explicit description of the monodromy of the trigonometric KZ equations, which is not quite correct as stated in [**12**, Thm. 3.3].[6]

A.1. Trigonometric KZ equations. Let A be a unital, associative algebra over $\mathbb{C}$ and $r \in A \otimes A$ a classical r–matrix, that is a solution of the classical Yang–Baxter equations

$$[r_{12}, r_{23}] + [r_{12}, r_{13}] + [r_{13}, r_{23}] = 0 \tag{A.1}$$

Set $r(u) = \dfrac{re^u + r_{21}}{e^u - 1}$, and let $s \in A$ be such that

$$[s \otimes 1 + 1 \otimes s, r] = 0 \tag{A.2}$$

[6]In turn, [**12**] amends the computation of the monodromy of the trigonometric KZ equations given in [**15**].

Let V be an A–module, $n \geq 1$, and $\mathbb{V}^{\otimes n}$ the trivial bundle over $\mathbb{C}^n$ with fibre $V^{\otimes n}$. The trigonometric KZ connection is the flat, $\mathfrak{S}_n$–equivariant connection on $\mathbb{V}^{\otimes n}$ given by

$$\nabla_{KZ} = d - \frac{\hbar}{2\pi\iota}\left(\sum_{i<j} r_{ij}(u_i - u_j)d(u_i - u_j) + \sum_i s^{(i)} du_i\right) \tag{A.3}$$

As explained in Section 5.2, its monodromy yields a representation

$$\pi_{KZ} : \Pi_n \to GL\left(V^{\otimes n}[[\hbar]]\right) \tag{A.4}$$

of the fundamental group Π_n of the configuration space of n points in $\mathbb{C}^\times$ on $V^{\otimes n}[[\hbar]]$.

A.2. Etingof–Kazhdan quantization. In order to describe the monodromy representation (A.4), one uses the machinery of Etingof–Kazhdan quantization [**13**]. Define the following finite–dimensional subspaces of A

$$\mathfrak{g}_+ = \{(1 \otimes f)(r) : f \in A^*\}$$
$$\mathfrak{g}_- = \{(g \otimes 1)(r) : g \in A^*\}$$

The following is a consequence of (A.1) (see [**13**, §5] for details)

PROPOSITION.

(1) $\mathfrak{g}_\pm$ *are Lie subalgebras of* A.
(2) *The following defines a non–degenerate pairing* $\mathfrak{g}_+ \otimes \mathfrak{g}_- \to \mathbb{C}$

$$\langle (1 \otimes f)(r), (g \otimes 1)(r) \rangle = (g \otimes f)(r)$$

(3) *The vector space* $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *is endowed with a unique Lie algebra structure extending those on* $\mathfrak{g}_\pm$ *and such that*

$$[x_+, x_-] = \mathrm{ad}^*(x_+)x_- - \mathrm{ad}^*(x_-)x_+$$

where $x_\pm \in \mathfrak{g}_\pm$ *and* ad^* *denotes the coadjoint action of* $\mathfrak{g}_\pm$ *on* $\mathfrak{g}_\mp \cong \mathfrak{g}_\pm^*$.
(4) *The map* $\pi : \mathfrak{g} \to A$ *whose restriction to* $\mathfrak{g}_\pm$ *is the canonical inclusion is a Lie algebra homomorphism.*
(5) *The canonical element* $1 \in \mathrm{End}(\mathfrak{g}_+) \cong \mathfrak{g}_+ \otimes \mathfrak{g}_-$ *maps to* r *under the homomorphism* $\pi : \mathfrak{g} \to A$.

It follows from Proposition A.2 that $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. Using the quantization theorem for finite–dimensional Manin triples [**13**, §3], we obtain a quasitriangular Hopf algebra $U_\hbar\mathfrak{g}$ with R–matrix $R \in U_\hbar\mathfrak{g}^{\otimes 2}$, and Hopf subalgebras $U_\hbar\mathfrak{g}_\pm \subset U_\hbar\mathfrak{g}$ in duality with each other, such that $R \in U_\hbar\mathfrak{g}_+ \otimes U_\hbar\mathfrak{g}_-$. Moreover, there is a canonical isomorphism of algebras $U_\hbar\mathfrak{g} \to U\mathfrak{g}[[\hbar]]$ which allows us to extend the map $\pi : \mathfrak{g} \to A$ to a homomorphism $U_\hbar\mathfrak{g} \to A[[\hbar]]$.

Consider now the following elements[7]

$$T = S \otimes \mathrm{id}(R_{21}) \in U_\hbar\mathfrak{g}^{\otimes 2} \tag{A.5}$$

$$C = m_{01}(T_{0n} \cdots T_{01}) = m_{01}(\mathrm{id} \otimes \Delta^{(n)}(T)) \in U_\hbar\mathfrak{g}^{\otimes n} \tag{A.6}$$

[7] The elements T, C differ slightly from those defined in [**12**] which are, respectively, $T' = \mathrm{id} \otimes S(R) = T_{21}$ and $C' = m_{01}(T_{01} \cdots T_{0n})$

where m_{01} is the multiplication on the first two copies in $U_\hbar\mathfrak{g}^{\otimes(n+1)}$, $\Delta^{(n)} : U_\hbar\mathfrak{g} \to U_\hbar\mathfrak{g}^{\otimes n}$ the iterated coproduct, and the second equality in (A.6) follows from the cabling identity $\Delta^{(n)} \otimes \mathrm{id}(R) = R_{0\,n}R_{1\,n}\cdots R_{n-1\,n}$ which implies that

$$\mathrm{id}\otimes\Delta^{(n)}(R_{21}) = (0\,1\cdots n)\Delta^{(n)} \otimes \mathrm{id}(R) = R_{1\,0}R_{2\,0}\cdots R_{n\,0}$$

Let $\mathcal{V}$ the $U_\hbar\mathfrak{g}$–module obtained from $V[[\hbar]]$ via the homomorphism $\pi : U_\hbar\mathfrak{g} \to A[[\hbar]]$. Then, the following holds

THEOREM. *The monodromy representation* π_{KZ} (A.4) *is equivalent to the action of* Π_n *on* $\mathcal{V}^{\otimes n}$ *given by*

$$b_i \mapsto (i\,i+1)\,R_{i,i+1}$$
$$\mathcal{X}_1 \mapsto (e^{\hbar s}u^{-1})^{(1)}C$$

where $u = m(S \otimes \mathrm{id})(R_{21}) \in U_\hbar\mathfrak{g}$ *is the Drinfeld element.*

The proof of this theorem is sketched in §A.3 – §A.5.

A.3. The first step towards the proof of Theorem A.2 is to relate the trigonometric KZ connection on n points (A.3) to the rational KZ connection on $n+1$ points. This is achieved by extending the Manin triple of Proposition A.2 to include a derivation.

Let $\rho_r = m(r_{21}) \in A$, so that if $r = \sum_i a_i\otimes b_i$, then $\rho_r = \sum_i b_i a_i$, and note that $[s, \rho_r] = 0$ by (A.2), Set $t = s + \rho_r$. It follows from (A.1) that $[t\otimes 1 + 1\otimes t, r] = 0$, which implies that $\mathrm{ad}(t)\mathfrak{g}_\pm \subset \mathfrak{g}_\pm$, and that $\mathrm{ad}(t)$ is a derivation of $\mathfrak{g}_\pm$ preserving the pairing between $\mathfrak{g}_+$ and $\mathfrak{g}_-$. Let $\mathfrak{g}' = (\mathfrak{g} \rtimes \mathbb{C}t) \oplus \mathbb{C}t^*$ be the extension of $\mathfrak{g} \rtimes \mathbb{C}t$ by a central element t^* determined by requiring that the commutator with t is the derivation $\mathrm{ad}(t)$ on $\mathfrak{g}$ and that, for $x, y \in \mathfrak{g}$

$$[x, y]_{\mathfrak{g}'} = [x, y]_{\mathfrak{g}} + ([t, x], y)\,t^*$$

Note that $\mathfrak{g}'$ is split over $\mathfrak{g}_\pm \rtimes \mathbb{C}t$. The inner product on $\mathfrak{g}$ extends to a non–degenerate, invariant bilinear form $(-,-)$ on $\mathfrak{g}'$ given by

$$(t, \mathfrak{g}) = (t^*, \mathfrak{g}) = 0 \qquad \text{and} \qquad (t, t^*) = 1$$

Thus, $(\mathfrak{g}', \mathfrak{g}'_+ = \mathfrak{g}_+ \rtimes \mathbb{C}t, \mathfrak{g}'_- = \mathfrak{g}_- \oplus \mathbb{C}t^*)$ is a Manin triple. The corresponding Lie cobracket $\delta_{\mathfrak{g}'} : \mathfrak{g}' \to \mathfrak{g}' \wedge \mathfrak{g}'$ is given by $\delta_{\mathfrak{g}'}(t) = \delta_{\mathfrak{g}'}(t^*) = 0$, $\delta_{\mathfrak{g}'}(x) = \delta_{\mathfrak{g}}(x)$ if $x \in \mathfrak{g}_+$ and

$$\delta_{\mathfrak{g}'}(x) = \delta_{\mathfrak{g}}(x) + [t, x] \wedge t^*$$

if $x \in \mathfrak{g}_-$. In particular, $\mathfrak{g}_+$ is a Lie subbialgebra of $\mathfrak{g}_+$, but $\mathfrak{g}_-$ is only a Lie subalgebra of $\mathfrak{g}'_-$.

Extend the algebra homomorphism $U\mathfrak{g} \to A$ to $\overline{U} = U\mathfrak{g}'/(t^*) \to A$ by $t \mapsto s + \rho_r$. Thus, V can be considered as a $\mathfrak{g}'$–module on which t^* acts trivially and t by $s + \rho_r$. Set

$$M_\pm = \mathrm{ind}^{\mathfrak{g}'}_{(\mathfrak{g}_\pm \rtimes \mathbb{C}t)\oplus\mathbb{C}t^*}\,\mathbb{C}_\pm$$

where $\mathfrak{g}_\pm \rtimes \mathbb{C}t$ acts on the one–dimensional module $\mathbb{C}_\pm$ by 0 and t^* as multiplication by ± 1. Frobenius reciprocity yields an isomorphism

$$\Xi : \mathrm{Hom}_{\mathfrak{g}_+\oplus\mathfrak{g}_-\oplus\mathbb{C}t^*}\left(M_+, M_-^*\widehat{\otimes}V^{\otimes n}\right) \to V^{\otimes n}$$

where $\widehat{\otimes}$ is the completed tensor product.

Consider now the following system of partial differential equations for a function $\Psi(z_0,\ldots,z_n)$ with values in $\mathrm{Hom}_{\mathfrak{g}_+\oplus\mathfrak{g}_-\oplus\mathbb{C}t^*}\left(M_+, M_-^*\widehat{\otimes}V^{\otimes n}\right)$

$$\frac{\partial\Psi}{\partial z_k}=\frac{\hbar}{2\pi\iota}\left(\sum_{j\neq k}\frac{\Omega'_{kj}}{z_k-z_j}\right)\Psi \tag{A.7}$$

where $\Omega'=\Omega+t\otimes t^*+t^*\otimes t$ is the Casimir tensor of $\mathfrak{g}'$. One readily checks that, for any $1\leq i,j\leq n$,

$$\Xi\,\Omega'_{ij}\,\Xi^{-1}=\Omega_{ij}\qquad\text{and}\qquad \Xi\,\Omega'_{0i}\,\Xi^{-1}=s^{(i)}-\sum_{\substack{1\leq k\leq n\\ k\neq i}}r_{ki}$$

Coupled with the change of variables $z_i=e^{u_i}$, $i=1,\ldots,n$, this yields the following

Proposition. *Under the Frobenius reciprocity isomorphism*

$$\Xi:\mathrm{Hom}_{\mathfrak{g}_+\oplus\mathfrak{g}_-\oplus\mathbb{C}t^*}\left(M_+, M_-^*\widehat{\otimes}V^{\otimes n}\right)\xrightarrow{\sim}V^{\otimes n}$$

the restriction of (A.7) *to* $z_0=0$ *coincides with* (A.3).

A.4. Denote the Etingof–Kazhdan quantization of a finite–dimensional Lie bialgebra $\mathfrak{l}$ by $U_\hbar\mathfrak{l}$. By functoriality of quantization,

$$U_\hbar((\mathfrak{g}_\pm\rtimes\mathbb{C}t)\oplus\mathbb{C}t^*)\cong(U_\hbar\mathfrak{g}_\pm\rtimes\mathbb{C}[t])\otimes\mathbb{C}[t^*]$$

Set $M^q_\pm=\mathrm{ind}_{U_\hbar((\mathfrak{g}_\pm\rtimes\mathbb{C}t)\oplus\mathbb{C}t^*)}^{U_\hbar\mathfrak{g}'}\mathbb{C}_\pm$, where $U_\hbar(\mathfrak{g}_\pm\rtimes\mathbb{C}t)$ acts trivially on $\mathbb{C}_\pm\cong\mathbb{C}$ and t^* as multiplication by ±1.

Regard $\mathcal{V}=V[[\hbar]]$ as a $U_\hbar\mathfrak{g}'$–module via the homomorphism $U_\hbar\mathfrak{g}'\cong U\mathfrak{g}'[[\hbar]]\to U\mathfrak{g}[[\hbar]]\to A[[\hbar]]$, where the intermediate map $\mathfrak{g}'\to\mathfrak{g}$ is given by $t^*\to0$ and $t\to s+\rho_r$. Let R' be the R–matrix of $U_\hbar\mathfrak{g}'$. The following is a consequence of the Kohno–Drinfeld theorem for $\mathfrak{g}'$ [**13**], together with Proposition A.3.

Proposition. *The monodromy representation* (A.4) *is equivalent to the representation of* Π_n *on* $\mathrm{Hom}_{U_\hbar(\mathfrak{g}_+\oplus\mathfrak{g}_-\oplus\mathbb{C}t^*)}\left(M^q_+,(M^q_-)^*\widehat{\otimes}\mathcal{V}^{\otimes n}\right)$ *given by*

$$b_i\mapsto(i\,i+1)R'_{i\,i+1}$$
$$\mathcal{X}_1\mapsto R'_{10}R'_{01}$$

where $(M^q_-)^*$ *is the right dual to* M^q_-.[8]

A.5. Since $R'_{i\,i+1}$ acts on $\mathcal{V}^{\otimes n}$ as $R_{i\,i+1}$, Proposition A.4 reduces the proof of Theorem A.2 to computing the action of $\Xi^q\,R'_{10}R'_{01}\,(\Xi^q)^{-1}$ on $V^{\otimes n}$, where Ξ^q is the isomorphism given by the composition

$$\begin{array}{ccc}\mathrm{Hom}_{U_\hbar(\mathfrak{g}_+\oplus\mathfrak{g}_-\oplus\mathbb{C}t^*)}\left(M^q_+,(M^q_-)^*\widehat{\otimes}\mathcal{V}^{\otimes n}\right) & \xrightarrow{\mathrm{ev}(\mathbf{1}^q_+)} & \left((M^q_-)^*\widehat{\otimes}\mathcal{V}^{\otimes n}\right)^{U_\hbar\mathfrak{g}_+}\\ \Big\downarrow{\scriptstyle\Xi^q} & & \Big\downarrow{\scriptstyle\phi}\\ \mathcal{V}^{\otimes n} & \xleftarrow[\mathrm{ev}(\mathbf{1}^q_-)]{} & \mathrm{Hom}_{U_\hbar\mathfrak{g}_+}\left(M^q_-,\mathcal{V}^{\otimes n}\right)\end{array} \tag{A.8}$$

[8]the action of $a\in U_\hbar\mathfrak{g}'$ on $\phi\in(M^q_-)^*$ is given by $a\phi=\phi\circ(S^{-1}a)$.

and ϕ is the restriction of the natural identification $(M_-^q)^* \otimes \mathcal{V}^{\otimes n} \cong \operatorname{Hom}\left(M_-^q, \mathcal{V}^{\otimes n}\right)$ to the subspace of $U_\hbar\mathfrak{g}_+$–invariant vectors.[9]

Write $R' = \alpha_j \otimes \beta^j$, where $\{\alpha_j\}$ is a basis of $U_\hbar\mathfrak{g}'_+$, $\{\beta^j\}$ is the dual basis of $U_\hbar\mathfrak{g}'_-$, and the summation over j is implicit. For a morphism

$$\Psi \in \operatorname{Hom}_{U_\hbar(\mathfrak{g}_+\oplus\mathfrak{g}_-\oplus\mathbb{C}t^*)}\left(M_+^q, (M_-^q)^* \widehat{\otimes} \mathcal{V}^{\otimes n}\right)$$

we compute $\mathcal{X}_1(\Xi^q(\Psi))$ in the following steps. In order to make the computations more transparent we abusively assume that $\Psi(\mathbf{1}_+^q)$ is an indecomposable tensor $m \otimes v_1 \otimes \cdots \otimes v_n$.

(a) The computation below corrects equation (5.4) of [**12**].

$$\begin{aligned}
\mathcal{X}_1(\Xi^q(\Psi)) &= \left\langle \mathbf{1}_-^q, R'_{10}R'_{01}\Psi\mathbf{1}_+^q \right\rangle \\
&= \left\langle \mathbf{1}_-^q, \beta^j\alpha_i \otimes \alpha_j\beta^i \otimes 1^{\otimes n-1}(\Psi\mathbf{1}_+^q)\right\rangle \\
&= \left\langle \mathbf{1}_-^q, \beta^j\alpha_i m\right\rangle \alpha_j\beta^i v_1 \otimes v_2 \otimes \cdots \otimes v_n \\
&= \left\langle S^{-1}(\beta^j)\mathbf{1}_-^q, \alpha_i m\right\rangle \alpha_j\beta^i v_1 \otimes v_2 \otimes \cdots \otimes v_n \\
&= \left\langle \mathbf{1}_-^q, \alpha_i m\right\rangle e^{\hbar t}\beta^i v_1 \otimes v_2 \otimes \cdots \otimes v_n \\
&= (e^{\hbar t})^{(1)}\, \Xi^q(R_{01}\Psi)
\end{aligned}$$

where we used the fact that t^* acts trivially on $\mathcal{V} \ni v_1$ and the fifth equality uses the fact that $\mathbf{1}_-^q$ is killed by $U_\hbar\mathfrak{g}_-$, that $t^*\mathbf{1}_-^q$ acts by -1 on M_-^q, and that the dual element to $(t^*)^k \in U_\hbar\mathfrak{g}'_-$ is $(\hbar t)^k/k! \in U_\hbar\mathfrak{g}'_+$.

(b) Write $R = a_j \otimes b^j$, where $\{a_j\}$ is a basis of $U_\hbar\mathfrak{g}_+$ and $\{b^j\}$ is the dual basis of $U_\hbar\mathfrak{g}_-$. Then

$$\begin{aligned}
\mathcal{X}_1(\Xi^q(\Psi)) &= \left\langle \mathbf{1}_-^q, a_i m\right\rangle e^{\hbar t}b^i v_1 \otimes v_2 \otimes \cdots \otimes v_n \\
&= (e^{\hbar t}b^i)^{(1)} \left\langle S^{-1}(a_i)\mathbf{1}_-^q, m\right\rangle v_1 \otimes v_2 \otimes \cdots \otimes v_n \\
&= (e^{\hbar t}b^i)^{(1)} \left\langle \mathbf{1}_-^q, m\right\rangle \Delta^{(n)}(S^{-1}(a_i))\,(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \\
&= (e^{\hbar t}b^i)^{(1)}\Delta^{(n)}(S^{-1}(a_i))\Xi^q(\Psi)
\end{aligned}$$

(c) Note that

$$\begin{aligned}
(b^i)^{(1)}\Delta^{(n)}(S^{-1}(a_i)) &= m_{01}\left(\operatorname{id}\otimes\Delta^{(n)} \circ \operatorname{id}\otimes S^{-1}(R_{21})\right) \\
&= m_{01}\left(\operatorname{id}\otimes\Delta^{(n)} \circ S \otimes \operatorname{id}(R_{21})\right) \\
&= C
\end{aligned}$$

where the second equality follows from the fact that $S \otimes S(R) = R$ and C is the element defined in (A.6)). It follows that $\mathcal{X}_1 \in \Pi_n$ acts on $\mathcal{V}^{\otimes n}$ as

$$\mathcal{X}_1 \mapsto (e^{\hbar s}w)^{(1)}C$$

where $w = e^{\hbar\rho_r}$ under the identification of $U_\hbar\mathfrak{g}$ with $U\mathfrak{g}[[\hbar]]$.

(d) To determine the element w, we restrict ourselves to the case $s = 0$ and $n = 1$. In this case the monodromy is trivial and hence we get

$$1 = w \cdot m_{01}(S \otimes \operatorname{id}(R_{21})) = wu$$

[9]This is the reason for considering the right dual of M_-^q instead of the left one as in [**12**]. For the latter, the natural identification $\left(M_-^q\right)^* \otimes \mathcal{V}^{\otimes n} \cong \operatorname{Hom}(M_-^q, \mathcal{V}^{\otimes n})$ does not restrict to the isomorphism between the subspace of $U_\hbar\mathfrak{g}_+$–invariant and $U_\hbar\mathfrak{g}_+$–linear morphisms.

where $u = S(b^i)a_i$ is the Drinfeld element. Thus $w = u^{-1}$, which completes the proof of Theorem A.2.

Appendix B. Proof of Proposition 9.10

We shall prove the result for $U_\hbar(L\mathfrak{sl}_2)$. The corresponding assertion for $U_\hbar(L\mathfrak{gl}_2)$ is proved similarly. The key step is to draw the following consequence of [**18**]

Lemma. *The following elements are in $\mathcal{J}^n$ for any $n \geq 0$ and $l > 0$*

$$H_{0;n} = H_0 + \frac{q-q^{-1}}{\hbar}\sum_{r=1}^{n}(-1)^r \begin{pmatrix} n \\ r \end{pmatrix} H_r$$

$$H_{l;n} = \frac{q-q^{-1}}{\hbar}\sum_{r=0}^{n}(-1)^r \begin{pmatrix} n \\ r \end{pmatrix} H_{l+r}$$

The proof of Lemma B will be given in §B.1–B.5.

Let us prove that $\widetilde{H}_r \in \mathcal{J}^r$ using Lemma B. We will need the following easy

Lemma. *Let $\{X_k\}_{k\in\mathbb{Z}}$ be elements of a vector space V. For each $t \in \mathbb{Z}$, and $0 \leq m \leq n$, define*

$$X_{n;t}^{(m)} = \sum_{s=0}^{n}(-1)^s \begin{pmatrix} n \\ s \end{pmatrix} s^m X_{s+t}$$

Then, for $m > 0$

$$X_{n;t}^{(m)} = -n\sum_{r=0}^{m-1} \begin{pmatrix} m-1 \\ r \end{pmatrix} X_{n-1;t+1}^{(r)}$$

In particular, $X_{n;t}^{(m)}$ can be written as a linear combination of $\{X_{n-k;t+k}^{(0)}\}_{1\leq k\leq m}$.

In particular, taking $X_0 = H_0$ and $X_k = \frac{q-q^{-1}}{\hbar}H_k$ for $k \neq 0$, we see that $X_{n;t}^{(m)} \in \mathcal{J}^{n-m}$ since, by Lemma B, $X_{n-k;t+k}^{(0)} \in \mathcal{J}^{n-k}$. Since multiplication by $\hbar$ maps $\mathcal{J}^n$ to $\mathcal{J}^{n+1}$, it follows that for any formal power series $p(u) \in 1 + u\mathbb{C}[[u]]$ the following expression lies in $\mathcal{J}^n$

$$X_{n;t}^{(p(u))} = \sum_{s=0}^{n}(-1)^s \begin{pmatrix} n \\ s \end{pmatrix} p(s\hbar)X_{s+t}$$

Taking $p(u) = \dfrac{u}{e^{u/2}-e^{-u/2}}$, so that $p(s\hbar) = \dfrac{\hbar}{q-q^{-1}}\dfrac{s}{[s]}$, we see that

$$\begin{aligned}\widetilde{H}_r &= H_0 + \sum_{s=1}^{r}(-1)^s \begin{pmatrix} r \\ s \end{pmatrix} \frac{s}{[s]}H_s \\ &= H_0 + \frac{q-q^{-1}}{\hbar}\left(\sum_{s=1}^{r}(-1)^s \begin{pmatrix} r \\ s \end{pmatrix} p(s\hbar)H_s\right)\end{aligned}$$

lies in $\mathcal{J}^r$, as claimed.

B.1. Some notation. For notational convenience, we set $x_k = \frac{q-q^{-1}}{\hbar} H_k$ for $k \geq 1$ and $x_0 = H_0$. Then we have

$$\psi^+(z) = q^{H_0} \exp\left(\hbar \sum_{r\geq 1} x_r z^{-r}\right)$$

and hence we obtain:

$$\psi_l = q^{H_0} \sum_{\lambda \vdash l} \hbar^{l(\lambda)} \frac{x_\lambda}{\prod_{i\geq 1} l_i!} \tag{B.1}$$

Set $y_{l;n} = q^{-H_0} \frac{q-q^{-1}}{\hbar} \phi_{l;n}$. Then using (B.1) we have:

$$y_{l;n} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \left(\sum_{\lambda \vdash l+r} \hbar^{l(\lambda)-1} \frac{x_\lambda}{\prod l_i!} \right) \tag{B.2}$$

$$y_{0;n} = \frac{1-e^{-\hbar x_0}}{\hbar} + \sum_{r=1}^{n} (-1)^r \binom{n}{r} \left(\sum_{\lambda \vdash r} \hbar^{l(\lambda)-1} \frac{x_\lambda}{\prod l_i!} \right) \tag{B.3}$$

B.2. It is clear from the definitions that $y_{l;n} \in \mathcal{J}^n$ for every $l, n \geq 0$. We denote by $p_{l;n}^{(m)}$ the coefficient of $\frac{\hbar^{m-1}}{m!}$ in $y_{l;n}$. Thus we have the following expressions (here $l \geq 1$):

$$p_{l;n}^{(m)} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \left(\sum_{\substack{\lambda \vdash l+r \\ l(\lambda)=m}} \frac{m! x_\lambda}{\prod l_i!} \right) \tag{B.4}$$

$$p_{0;n}^{(m)} = (-1)^{m-1} x_0^m + \sum_{r=1}^{n} (-1)^r \binom{n}{r} \left(\sum_{\substack{\lambda \vdash r \\ l(\lambda)=m}} \frac{m! x_\lambda}{\prod l_i!} \right) \tag{B.5}$$

We will prove the following stronger version of Lemma B.

LEMMA. *For each $l, n \geq 0$ and $m \geq 1$ we have:*

$$p_{l;n}^{(m)} \in \mathcal{J}^{n-m+1}$$

Note that the assertion of Lemma B is $m = 1$ case of that of Lemma B.2.

B.3. Proof of Lemma B.2. We begin by considering $l \geq 1$ case. In this case we have the following relation for $m \geq 2$:

$$p_{l;n}^{(m)} = \sum_{t=1}^{l-1} p_{t;0}^{(m-1)} p_{l-t;n}^{(1)} - \sum_{k=0}^{n-1} p_{1;k}^{(1)} p_{l;n-k-1}^{(m-1)} \tag{B.6}$$

We prove $p_{l;n}^{(m)} \in \mathcal{J}^{n-m+1}$ for every $l \geq 1$, $m \geq 1$, $n \geq 1$ by induction on n and l in the following manner. Consider the base case of $n = 1$:

$$y_{l;1} = p_{l;1}^{(1)} + O(\hbar) \in \mathcal{J}$$

which implies that $p_{l;1}^{(1)} \in \mathcal{J}$. For $n = 1$ and $m \geq 2$ the statement is vacuous. Thus we have proved the assertion for $n = 1$ and all $l, m \geq 1$.

Now we proceed to the induction step. Let us assume that $p^{(m)}_{l;n'} \in \mathcal{J}^{n'-m+1}$ for every $n' < n$ and $l, m \geq 1$. Now the same assertion for $n' = n$ is proved for $m \geq 2$ by using (B.6) and induction on l. The base case $l = 1$ is established by (B.6):

$$p^{(m)}_{1;n} = -\sum_{k=1}^{n-1} p^{(1)}_{1;k} p^{(m-1)}_{1;n-k-1}$$

since all the terms on the right–hand side have smaller n. Proceeding by induction on l we can prove the desired assertion for $n' = n$ and for every $m \geq 2$, $l \geq 1$. The case $m = 1$ follows from the fact that

$$y_{l;n} = p^{(1)}_{l;n} + \sum_{m\geq 2} \frac{\hbar^{m-1}}{m!} p^{(m)}_{l;n} \in \mathcal{J}^n$$

B.4. Next we consider the $l = 0$ case. In this case we will need the following relation (again for $m \geq 2$):

$$p^{(m)}_{0;n} = (-1)^{m-1} x_0^{m-1} p^{(1)}_{0;n-1} - \sum_{k=0}^{n-2} p^{(1)}_{1;k} p^{(m-1)}_{0;n-1-k} \tag{B.7}$$

Again $p^{(m)}_{0;n} \in \mathcal{J}^{n-m+1}$ is proved by an induction argument, similar to the one given above, using (B.7), combined with the result of the previous section.

B.5. Proof of (B.6) **and** (B.7)**.** The proofs of relations (B.6) and (B.7) are similar. We provide the main steps in the proof of (B.6) and leave a few straightforward checks to the reader.

For the proof, it will be convenient to write $p^{(m)}_{l;n}$ as:

$$p^{(m)}_{l;n} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \left(\sum_{\substack{a_1,\ldots,a_m \geq 1 \\ a_1+\cdots+a_m = r+l}} x_{a_1} \ldots x_{a_m} \right)$$

which implies the following verification:

$$\sum_{t=1}^{l-1} p^{(m-1)}_{t;0} p^{(1)}_{l-t;n} - p^{(m)}_{l;n} = \sum_{s=0}^{n-1} (-1)^s \binom{n}{s+1} \left(\sum_{\substack{a_1,\ldots,a_{m-1} \geq 1 \\ l \leq a_1+\cdots+a_{m-1} \leq l+s}} x_{a_1} \cdots x_{a_{m-1}} \right) \tag{B.8}$$

Let us denote the expression obtained above by $g_{m,l}(n)$. Recall that the equation (B.6) is equivalent to

$$g_{m,l}(n) = \sum_{k=0}^{n-1} p^{(1)}_{1;k} p^{(m-1)}_{l;n-k-a}$$

This equation can be verified in the following manner. It is easy to check that the claimed equation holds for $n = 1$. Moreover both sides satisfy the following recurrence relation:

$$F_{m,l}(n+1) - F_{m,l}(n) + F_{m,l+1}(n) = p^{(1)}_{1;n} p^{(m-1)}_{l;0}$$

which implies the desired assertion by induction on n.

References

[1] P. Baumann, *The q–Weyl group of a q–Schur algebra*, preprint, 1999.

[2] Jonathan Beck, *Braid group action and quantum affine algebras*, Comm. Math. Phys. **165** (1994), no. 3, 555–568. MR1301623 (95i:17011)

[3] Joan S. Birman, *Braids, links, and mapping class groups*, Princeton University Press, Princeton, N.J., 1974. Annals of Mathematics Studies, No. 82. MR0375281 (51 #11477)

[4] Jonathan Brundan and Alexander Kleshchev, *Parabolic presentations of the Yangian $Y(\mathfrak{gl}_n)$*, Comm. Math. Phys. **254** (2005), no. 1, 191–220, DOI 10.1007/s00220-004-1249-6. MR2116743 (2005j:17012)

[5] Vyjayanthi Chari and Andrew Pressley, *Yangians and R-matrices*, Enseign. Math. (2) **36** (1990), no. 3-4, 267–302. MR1096420 (92h:17009)

[6] Vyjayanthi Chari and Andrew Pressley, *Quantum affine algebras*, Comm. Math. Phys. **142** (1991), no. 2, 261–283. MR1137064 (93d:17017)

[7] Ivan Cherednik, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series, vol. 319, Cambridge University Press, Cambridge, 2005. MR2133033 (2007e:32012)

[8] Jin Tai Ding and Igor B. Frenkel, *Isomorphism of two realizations of quantum affine algebra $U_q(\mathfrak{gl}(n))$*, Comm. Math. Phys. **156** (1993), no. 2, 277–300. MR1233847 (94i:17020)

[9] V. G. Drinfel′d, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR **283** (1985), no. 5, 1060–1064 (Russian). MR802128 (87h:58080)

[10] V. G. Drinfel′d, *A new realization of Yangians and of quantum affine algebras*, Dokl. Akad. Nauk SSSR **296** (1987), no. 1, 13–17 (Russian); English transl., Soviet Math. Dokl. **36** (1988), no. 2, 212–216. MR914215 (88j:17020)

[11] V. G. Drinfel′d, *Almost cocommutative Hopf algebras*, Algebra i Analiz **1** (1989), no. 2, 30–46 (Russian); English transl., Leningrad Math. J. **1** (1990), no. 2, 321–342. MR1025154 (91b:16046)

[12] Pavel Etingof and Nathan Geer, *Monodromy of trigonometric KZ equations*, Int. Math. Res. Not. IMRN **24** (2007), Art. ID rnm123, 15, DOI 10.1093/imrn/rnm123. MR2377010 (2008k:32039)

[13] Pavel Etingof and David Kazhdan, *Quantization of Lie bialgebras. I*, Selecta Math. (N.S.) **2** (1996), no. 1, 1–41, DOI 10.1007/BF01587938. MR1403351 (97f:17014)

[14] Pavel Etingof and David Kazhdan, *Quantization of Lie bialgebras. VI. Quantization of generalized Kac-Moody algebras*, Transform. Groups **13** (2008), no. 3-4, 527–539, DOI 10.1007/s00031-008-9029-6. MR2452604 (2010d:17016)

[15] Pavel Etingof and Olivier Schiffmann, *Lectures on quantum groups*, Lectures in Mathematical Physics, International Press, Boston, MA, 1998. MR1698405 (2000e:17016)

[16] S. Gautam and V. Toledano Laredo, *Yangians and quantum loop algebras*, Selecta Math. (N.S.), 2012, DOI 10.1007/s00029-012-0114-2, published online, 66 pages, arXiv:1012.3687.

[17] ———, in preparation.

[18] N. Guay and X. Ma, *From quantum loop algebras to Yangians*, J. Lond. Math. Soc., 2012, DOI 10.1112/jlms/jds021, published online.

[19] A. N. Kirillov and N. Reshetikhin, *q-Weyl group and a multiplicative formula for universal R-matrices*, Comm. Math. Phys. **134** (1990), no. 2, 421–431. MR1081014 (92c:17023)

[20] George Lusztig, *Introduction to quantum groups*, Progress in Mathematics, vol. 110, Birkhäuser Boston Inc., Boston, MA, 1993. MR1227098 (94m:17016)

[21] I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Mathematics, vol. 157, Cambridge University Press, Cambridge, 2003. MR1976581 (2005b:33021)

[22] Alexander Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, vol. 143, American Mathematical Society, Providence, RI, 2007. MR2355506 (2008m:17033)

[23] Nguyễn Viêt Dũng, *The fundamental groups of the spaces of regular orbits of the affine Weyl groups*, Topology **22** (1983), no. 4, 425–435, DOI 10.1016/0040-9383(83)90035-6. MR715248 (85f:57001)

[24] Ya. S. Soĭbel′man, *Algebra of functions on a compact quantum group and its representations*, Algebra i Analiz **2** (1990), no. 1, 190–212 (Russian); English transl., Leningrad Math. J. **2** (1991), no. 1, 161–178. MR1049910 (91i:58053a)

[25] V. Tarasov and A. Varchenko, *Duality for Knizhnik-Zamolodchikov and dynamical equations*, Acta Appl. Math. **73** (2002), no. 1-2, 141–154, DOI 10.1023/A:1019787006990. The 2000 Twente Conference on Lie Groups (Enschede). MR1926498 (2003h:17024)

[26] Valerio Toledano Laredo, *A Kohno-Drinfeld theorem for quantum Weyl groups*, Duke Math. J. **112** (2002), no. 3, 421–451, DOI 10.1215/S0012-9074-02-11232-0. MR1896470 (2003m:17016)

[27] Valerio Toledano Laredo, *Quasi–Coxeter algebras, Dynkin diagram cohomology and quantum Weyl groups*, Int. Math. Res. Pap. **2008** (2008), 167 pp.

[28] Valerio Toledano Laredo, *The trigonometric Casimir connection of a simple Lie algebra*, J. Algebra **329** (2011), 286–327, DOI 10.1016/j.jalgebra.2010.05.025. MR2769327 (2012c:17020)

[29] Harm van der Lek, *Extended Artin groups*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 117–121. MR713240 (85b:14005)

[30] Harm van der Lek, *The homotopy type of hyperplane complements*, Ph.D. thesis, Katholieke Universiteit Nijmegen, 1983.

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